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COMPETING EXPONENTIAL RISKS, WITH PARTICULAR REFERENCE TO THE STUDY OF SMOKING AND LUNG CANCER*

JOSEPH BERKSON
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AND
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In various contexts of statistical problems one deals with the measurement of a "risk"—a risk of dying, a risk of becoming ill, and so forth. In applied vital statistics various indices are used to measure such risks, and typically these are *not* probabilities. In formal mathematical statistical analyses, typically they *are* measured as probabilities. A special problem arises when more than one risk must be measured at the same time. Even if they are independent in a technical statistical sense, the presence of one risk will complicate the measurement of another. This situation has been referred to as "competing risks." In this paper we apply an analysis of competing risks to some questions that arose in the study of the relation of smoking to lung cancer.

INTRODUCTION

IN SOME "prospective" studies, in which the death rates from various causes among smokers were compared with the similar death rates among nonsmokers, the "mortality ratio," which, in the present application, is the ratio of the death rate among smokers to the corresponding death rate among nonsmokers, was used to measure the "association" between smoking and death rates. The use of this index, in the way it had been employed in these investigations, was criticized by one of the present authors [1] as misleading. Instead, the simple difference of the death rates as between smokers and nonsmokers was used for the analysis of the primary questions presenting themselves.[1]

Dr. Mindel Sheps, in a commentary article [17], extended the case against the mortality ratio. At the same time she criticized the use of the simple difference of the rates as illogical.[2] On the basis of a heuristic development involving the idea of persons "exposed to risk," she presented a different computation to

* This investigation was supported in part by a grant from the National Science Foundation.

† Now at The Public Health Research Institute of the City of New York, Inc.

[1] The difference of rates was used for analyses also by some statisticians reporting prospective studies, in particular, Hammond and Horn [12].

[2] She says, " . . . the difference . . . has little meaning unless it is related to the number of potential survivors." This stricture is rather too severe. Sheps herself says (page 1211) "The absolute difference . . . describes concretely the number of persons affected. . . "

measure the putative effect of smoking, and later [16] showed this to be a maximum likelihood estimate. Sheps dealt only with total deaths from all causes, and not with deaths from specific causes. It was to be noted that the formula she used had appeared in various statistical contexts, though it had not previously been developed as a maximum likelihood estimate.

It appeared useful to develop similar formulas applicable to deaths from specific causes. Here again it was to be observed that, at least for one element of the problem, some relevant formulas had been developed, in this case by Neyman and associates, under the concept of "competing risks." It occurred to us that it would be worthwhile systematically to elaborate the various formulas in a unified way, basing the development explicitly on the concept of competing risks.

The first part of the presentation develops the necessary formulas, in a standard way. The motive is expository. A considerable number of the formulas have already been published, and are to be found in the references listed at the end of the paper.

Following this is presented a model for the effect of smoking in causing death from a specific disease, in which the parametric measure of the effect is in terms of the net probability of death from this disease. An example is given, and some comments are made, contrasting this measure with the mortality ratio.

COMPETING RISKS

We consider the model of exposure to two competing risks in the sense of Neyman [15] and take first a very simple example.

Two marksmen shoot at a range of targets under conditions in which, if a target is struck, it instantly drops from view so that it cannot be struck again. Represent the striking rate of marksman 1, that is the probability of a hit when he is firing alone as q_1, and similarly the rate of marksman 2 when he is firing alone as q_2. The probability when one risk operates alone is called the net risk or rate and is represented by lower case q; when it operates together with another risk it is called the crude risk or rate and is represented by capital Q.

Suppose N targets are exposed and marksman 1 shoots first, followed by marksman 2:

$$\text{Rate for 1 is } Q_1 = q_1 \tag{1}$$

$$\text{Rate for 2 is } Q_2 = (1 - q_1)q_2 \tag{2}$$

$$\text{Total rate is } Q = Q_1 + Q_2 = q_1 + q_2 - q_1q_2 \tag{3}$$

Suppose marksman 2 shoots first, followed by marksman 1, then:

$$\text{Rate for 2 is } Q_2 = q_2 \tag{4}$$

$$\text{Rate for 1 is } Q_1 = (1 - q_2)q_1 \tag{5}$$

$$\text{Total rate is } Q = q_1 + q_2 - q_1q_2 \tag{6}$$

It is seen that the total crude rate with both marksmen shooting is the same, whichever marksman shoots first, and assuming independence of the net probabilities q_1 and q_2 this will be true in general. Regardless of the order of shooting

or whether the two marksmen shoot together, the total crude rate is given by (3), (6), which, of course, can be derived as the complement of the product of the probabilities, $p_1 = 1 - q_1$, and $p_2 = 1 - q_2$, of not being struck (survival rate).

If, from independent trial, we know q_1, the net rate of marksman 1, and have a record of Q, the crude rate when both shot together, we can derive the net rate q_2 from (3):

$$q_2 = \frac{Q - q_1}{1 - q_1} \tag{7}$$

But, suppose we did not independently know the net rate q_1 or q_2, and knew from observation only the crude rates Q_1, Q_2 and $Q = Q_1 + Q_2$, then we could not determine the net rates q_1 and q_2 from these crude rates alone. We have seen that, with the same net rates q_1 and q_2, although the total crude rate Q is independent of the order of shooting, the crude rates Q_1, Q_2 for each marksman will depend on which marksman shot first. This problem of estimating a risk, when another independent risk is operating at the same time, is one that arises in different contexts in many statistical problems.

THE INSTANTANEOUS RISK, THE NET RATE, AND THE CRUDE DATE

In order to estimate the net q's from the crude Q's, something has to be known regarding the time relation of the risks. A simplifying assumption, which is frequently reasonable, is to suppose that each instantaneous risk[3] (called the "force of mortality" in actuarial texts) is constant over the period of observation, and that the risks are independent. We will use β to represent the instantaneous risk, and shift to the example of dealing with two mortality rates, namely q_1, the net mortality from some specified disease, and q_2, the net mortality from all other diseases than 1, taken together and considered as a single risk. Then the net probability of death before t from the respective causes is given by[4]

$$q_{1t} = 1 - e^{-\beta_1 t} \tag{8}$$

$$q_{2t} = 1 - e^{-\beta_2 t} \tag{9}$$

where β_1 is the instantaneous risk for net risk 1, β_2 is the instantaneous risk for net risk 2, and t is the time measured from $t = 0$. From (8), (9) we have the corresponding net probability of survival to time t

$$p_{1t} = 1 - q_{1t} = e^{-\beta_1 t} \tag{10}$$

$$p_{2t} = 1 - q_{2t} = e^{-\beta_2 t} \tag{11}$$

The probability of survival to time t, with both risks operating together, is the product of (10), (11)

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[3] If l is the number surviving at time t,

$$-\frac{dl_t}{l_t dt} = -\frac{d(\ln l_t)}{dt}$$

is the instantaneous risk at t.

[4] In some situations the known facts contradict the assumption of exponentiality of the net risks, as for instance the consideration of "withdrawals" as a competing risk in computing the T-year survival rate [4]. For the present purpose of computing relative competing risks among different independent risks, at least when the risks are small, the assumption of a different function, instead of the exponential, makes little difference in the results.

$$P_t = e^{-(\beta_1+\beta)t} = e^{-\beta t} \tag{12}$$

and the probability of dying before t is

$$Q_t = 1 - P_t = 1 - e^{-\beta t} \tag{13}$$

where $\beta = \beta_1 + \beta_2$.

The formulas (10), (11), (12) represent "survival functions" in the context of actuarial discussions.

It is seen, comparing (12) and (10), (11), that the crude rate with several net risks operating together has the same exponential form that defines the net rates, with the instantaneous crude risk equal to the sum of the instantaneous net risks. As has been remarked previously, some functional relationship of the risk with time is needed for estimating the q's if we are given only a knowledge of the Q's. The exponential function is unique in that the function remains the same when a new risk is added (or removed).

Without loss of generality, we can consider the period of observation as from $t=0$ to $t=1$ $(0 \leqq t \leqq 1)$. If observations are made over a period from $t=0$ to $t=T$, we can utilize the formulas given here by replacing the scale t with a scale x, where $x = t/T$.

The proportion of persons dying from cause 1 over the unit period from $t=0$ to $t=1$, which is the crude death rate from cause 1, is

$$Q_1 = \int_0^1 e^{-\beta t}\beta_1 dt = \frac{\beta_1}{\beta}\left[1 - e^{-\beta}\right] = \frac{\beta_1}{\beta} Q \tag{14}$$

and similarly for cause 2

$$Q_2 = \frac{\beta_2}{\beta}\left[1 - e^{-\beta}\right] = \frac{\beta_2}{\beta} Q \tag{15}$$

and for all causes, the probability of death before the end of the period is

$$Q = Q_1 + Q_2 = 1 - e^{-\beta} \tag{16}$$

and the probability of survival to the end of the period is

$$P = 1 - Q = e^{-\beta} \tag{17}$$

The net death rates over the unit period are

$$q_1 = 1 - p_1 = 1 - e^{-\beta_1} \tag{18}$$

$$q_2 = 1 - p_2 = 1 - e^{-\beta_2} \tag{19}$$

If, observing the crude rates Q_1, Q_2 and $Q = Q_1 + Q_2$, we wish the net rates q_1, q_2, these can be obtained from[5]

$$\ln (1 - q_1) = -\beta_1 = \frac{Q_1}{Q} \ln (1 - Q) \tag{20}$$

[5] Suppose that the observed death rate per annum from all causes, computed as a probability, is Q, and the observed rate from some specific cause, say lung cancer, is Q_1. We wish to know the total death rate if lung cancer as a cause of death were eliminated. We may consider the death rate from all causes except lung cancer as a single death rate $Q_1 = Q - Q_1$. The desired rate is then obtained as q_2 from (21). This is equivalent to the formula given by Greville [8] to compute the life table resulting from the elimination of specific causes of death.

$$\ln (1 - q_2) = -\beta_2 = \frac{Q_2}{Q} \ln (1 - Q) \tag{21}$$

which are derived directly from (14), (15), (17), (18), (19).

MAXIMUM LIKELIHOOD ESTIMATE, TIME ESTIMATES

We will now consider that we are dealing with observations subject to random variation, and develop the maximum likelihood estimates and variances of the instantaneous risks and the net rates.

Suppose we observe not only the number of deaths, but also the time when each death occurred. N individuals are observed over the unit period from $t=0$ to $t=1$. We have observed d_1 deaths at times $t_{11}, t_{12}, \cdots, t_{1d_1}$, and d_2 deaths at times $t_{21}, t_{22}, \cdots, t_{2d_2}$, and $s=N-d_1-d_2$ survivors to the end of the unit period. A death from cause 1 at time t_{1i} contributes to the joint sample probability a factor $\beta_1 e^{-\beta t_{1i}}$, where $\beta=\beta_1+\beta_2$. Similarly a death from cause 2 at time t_{2j} contributes a factor $\beta_2 e^{-\beta t_{2j}}$. Each survivor contributes a factor $e^{-\beta}$. The joint probability is proportional to

$$\begin{aligned} \phi &= \beta_1^{d_1} e^{-\beta \Sigma t_1} \beta_2^{d_2} e^{-\beta \Sigma t_2} e^{-\beta s} \\ &= \beta_1^{d_1} e^{-\beta_1 (\Sigma t + s)} \beta_2^{d_2} e^{-\beta_2 (\Sigma t + s)}. \end{aligned} \tag{22}$$

From (22) it is to be observed that $[d_1, (\Sigma t+s)]$ is sufficient for β_1 (with β_2 fixed), and $[d_2, (\Sigma t+s)]$ is sufficient for β_2 (with β_1 fixed). It is of interest to note that for β_1, with d_1 and s, it is Σt, not Σt_1 which is sufficient; similarly for β_2, with d_2 and s, it is Σt, not Σt_2 which is sufficient. In noting the time of death, it is irrelevant for the estimation of β_1, β_2, whether the death is from cause 1 or cause 2.

From the joint probability of the set of observations (22), the maximum likelihood estimates and their asymptotic variances are obtained in the standard way as derived from Fisher [6], [7]. The first derivatives $\partial \ln \phi/\partial\beta_1$ and $\partial \ln \phi/\partial\beta_2$ are set equal to zero, and the solutions for β_1 and β_2 are the respective maximum likelihood estimates. The asymptotic variances are obtained from the inverse of the matrix of terms

$$\left\{-E \frac{\partial^2 \ln \phi}{\partial\beta_i \partial\beta_j}\right\}$$

$$\hat{\beta}_1 = \frac{d_1}{\Sigma t + s}, \qquad \hat{\beta}_2 = \frac{d_2}{\Sigma t + s} \tag{23}$$

where $\Sigma t = \Sigma t_1 + \Sigma t_2$

$$\sigma^2_{\hat{\beta}_1} \text{ (asymptotic)} = \frac{\beta_1^2 + \beta_1\beta_2}{N(1 - e^{-\beta})} \tag{24}$$

$$\sigma^2_{\hat{\beta}_2} \text{ (asymptotic)} = \frac{\beta_2^2 + \beta_1\beta_2}{N(1 - e^{-\beta})} \tag{25}$$

For the estimate of any of the probabilities q, we can remember that, $q=1-e^{-\beta}$, $\hat{q}=1-e^{-\hat{\beta}}$, var. $\hat{q}=(1-q)^2$ var. $\hat{\beta}=p^2$ var. $\hat{\beta}$.

The maximum likelihood estimates (23) for β_1, β_2, are functions of the sufficient statistics for the respective parameters, following a general theorem for maximum likelihood estimates. However, the maximum likelihood estimates, in this case, themselves are not sufficient, since it is possible to have two samples with the same maximum likelihood estimates, though the values of their sufficient statistics are not identical.

If there is no particular interest in the mortality from specific diseases, but only in the total force of mortality $\beta=\beta_1+\beta_2$, then only the number of deaths, regardless of cause of death, and their times are required. The maximum likelihood estimate for $\beta=\beta_1+\beta_2$, and its asymptotic variance are [5], [9], [14]:

$$\hat{\beta} = \frac{d}{\Sigma t + s} \tag{26}$$

$$\sigma^2 \text{ (asymptotic)} = \frac{\beta^2}{N(1 - e^{-\beta})} \tag{27}$$

where $d=d_1+d_2$.

The statistic $[d, (\Sigma t+s)]$ is sufficient for β, but, as with estimation of β_1, β_2, the maximum likelihood estimate is not sufficient. If, instead of truncating the experience at some time T, it is truncated when some fixed number $r \leqq N$ of deaths is observed, then the maximum likelihood estimate is sufficient [5]. It is to be noted that

$$\hat{\beta} = \hat{\beta}_1 + \hat{\beta}_2,$$

and this exemplifies the generality that the maximum likelihood estimate of a function is the function of the maximum likelihood estimate. Also

$$\hat{\beta}_1 = \frac{d_1}{d}\hat{\beta}, \qquad \hat{\beta}_2 = \frac{d_2}{d}\beta. \tag{28}$$

So with β estimated from (26), $\hat{\beta}_1$, $\hat{\beta}_2$, can be estimated as in proportion to the respective number of deaths d_1 and d_2, without computation from (23).

FREQUENCY ESTIMATES

We may now turn to the situation in which we have observed the number of deaths d_1 and d_2 from each respective cause, but not the times of death, and develop the maximum likelihood estimates based on the observed frequencies. The sample size $=N$, $d=d_1+d_2$ is the total number of deaths, and $s=N-d_1-d_2$ is the number of survivors. We will recall that the probability of a death from cause 1 in the unit period is

$$Q_1 = 1 - P_1 = \frac{\beta_1}{\beta}(1 - e^{-\beta}) \tag{29}$$

and for a death from cause 2 it is

$$Q_2 = 1 - P_2 = \frac{\beta_2}{\beta}(1 - e^{-\beta}) \tag{30}$$

and the probability of survival to the end of the unit period is

$$P = e^{-\beta} \tag{31}$$

The probability of the total sample is proportional to

$$\phi = \left[\frac{\beta_1}{\beta}(1 - e^{-\beta})\right]^{d_1}\left[\frac{\beta_2}{\beta}(1 - e^{-\beta})\right]^{d_2} e^{-\beta s} \tag{32}$$

From (32) the maximum likelihood estimates and their variances are

$$\hat{\beta}_1 = \frac{d_1}{d} \ln \frac{N}{s} \tag{33}$$

$$\hat{\beta}_2 = \frac{d_2}{d} \ln \frac{N}{s} \tag{34}$$

$$\sigma^2_{\hat{\beta}_1} \text{ (asymptotic)} = \frac{1}{N}\left[\frac{\beta_1\beta_2}{1 - e^{-\beta}} + \frac{\beta_1^2(1 - e^{-\beta})}{\beta^2 e^{-\beta}}\right] \tag{35}$$

$$\sigma^2_{\hat{\beta}_2} \text{ (asymptotic)} = \frac{1}{N}\left[\frac{\beta_1\beta_2}{1 - e^{-\beta}} + \frac{\beta_2^2(1 - e^{-\beta})}{\beta^2 e^{-\beta}}\right] \tag{36}$$

Since, in applying (20), (21) to obtain β_1, β_2, one takes for Q_1, Q_2 and Q, the corresponding relative frequencies d_1/N, d_2/N, and $d/N = 1 - s/N$, the maximum likelihood estimates (33), (34) are seen, in effect, to be the same as (20), (21) which were developed directly.

For the estimate of β, we write the joint probability as proportional to

$$\phi = (1 - e^{-\beta})^d e^{-\beta s}. \tag{37}$$

From (37) we derive the maximum likelihood estimate and its variance

$$\hat{\beta} = \ln (N/s) \tag{38}$$

$$\sigma^2_{\hat{\beta}} \text{ (asymptotic)} = \frac{1 - e^{-\beta}}{Ne^{-\beta}} \tag{39}$$

As with the time estimates

$$\hat{\beta} = \hat{\beta}_1 + \hat{\beta}_2$$

The probability of death in the period is

$$Q = 1 - P = 1 - e^{-\beta} \tag{40}$$

and therefore the variance (39) can be written

$$\sigma^2_{\hat{\beta}} \text{ (asymptotic)} = \frac{Q}{NP}. \tag{41}$$

If we think of estimating $Q = 1 - e^{-\beta}$ instead of β, then

$$\hat{Q} = d/N \tag{42}$$

$$\sigma^2_{\hat{Q}} = \frac{PQ}{N} \tag{43}$$

In the case of (43) the variance is exact instead of only asymptotic, (42) being the usual estimate of a binomial parameter.

COMPARISON OF VARIANCES

We have presented two sets of estimates, one based on the number of observed deaths and the times of these deaths—the "time estimate," and the other based only on the number of deaths—the "frequency estimate." Presumably, the time estimate contains more "information" in regard to β, and this should be reflected in a smaller variance of the estimate. We have only formulas for the asymptotic variances, but we believe that, in the present situation, they probably are very good approximations of the exact variance for moderately large samples. Since $\beta = \beta_1 + \beta_2$, it seems reasonable to assume that a comparison of the estimates for β should give a fair idea of how they compare for estimates of β_1, β_2.

It is clear on inspection that the frequency estimate (38)

$$\hat{\beta} = \ln (N/s)$$

cannot be good for large β, when the Q is close to 1. In this case the probability that $s=0$, for even fairly large N, will not be small, and for all samples with $s=0$ the estimate of β from (38) is not determinable.

In Table 1 are shown the relative variances of the two estimates for different values of Q. It is seen that for small $Q \leqq .05$ the variance of the time estimate is virtually equal to that of the frequency estimate. For $Q \leqq 0.6$ the relative efficiency of the frequency estimate is greater than 0.9. Only with $Q > 0.9$ does the efficiency fall below 0.5. Since the frequency estimate requires only the number of deaths and not their times and is easier to compute than the time estimate, it may be found satisfactory for use, except with very large Q.

TABLE 1. COMPARISON OF VARIANCES. TIME ESTIMATE AND FREQUENCY ESTIMATE

Q	N times the asymptotic variance		Relative efficiency
	Time estimate	Frequency estimate	
.001	.0010	.0010	1.000
.01	.0101	.0101	1.000
.05	.0526	.0526	1.000
.10	.1110	.1111	.999
.50	.9609	1.0000	.961
.60	1.3993	1.5000	.933
.90	5.8910	9.0000	.655
.95	8.9744	19.0000	.472
.99	21.4218	99.0000	.216

MEASURE OF THE MORTAL EFFECT OF SMOKING

Sheps [16] has considered how the effect of smoking may be assessed from the analysis of "prospective" studies of the relation of smoking to death rates. On the basis of a heuristic development involving the idea of "exposed to risk," she derived the maximum likelihood frequency estimate of the net risk, due to smoking, of death from all causes. For this particular estimate, no assumption is required regarding the time relation of the risks. We will consider the problem more in detail, particularly in regard to death from specific causes, on the basis of the development presented in the previous sections.

As before, consider deaths as segregated in two classes, those due to some specific disease for which we take cancer of the lung as an example, and all other causes taken together and considered as a single cause. Nonsmokers are subject to deaths from "natural causes." Smokers also are subject to death from natural causes but we assume that, in addition, they are subject to deaths from lung cancer caused by a specific carcinogen Y, and from other diseases caused by substances X, the substances Y and X being contained in tobacco smoke. We assume that these causes act independently, and that the net probabilities of death before t $(0 \leqq t \leqq 1)$ in a unit period are given by

$$q'_{1t} = 1 - e^{-\beta'_1 t} \tag{44}$$

$$q'_{2t} = 1 - e^{-\beta'_2 t} \tag{45}$$

$$q_{1t} = 1 - e^{-\beta_1 t} \tag{46}$$

$$q_{2t} = 1 - e^{-\beta_2 t} \tag{47}$$

where q'_{1t}, q'_{2t} refer to net probabilities of death due to "natural causes"—from lung cancer and other diseases respectively, and q_{1t}, q_{2t} refer to net probabilities of death due to substances Y and X contained in tobacco smoke—from lung cancer and other diseases respectively.

Then, for the nonsmokers, the observed crude probabilities of death from lung cancer, other diseases, and all causes respectively are

$$Q'_{1t} = \frac{\beta'_1}{\beta'}\left[1 - e^{-\beta' t}\right] \tag{48}$$

$$Q'_{2t} = \frac{\beta'_2}{\beta'}\left[1 - e^{-\beta' t}\right] \tag{49}$$

$$Q'_t = 1 - e^{-\beta' t} \tag{50}$$

where $\beta' = \beta'_1 + \beta'_2$.

In the case of the smokers, there will be an increment of β_1 and β_2 (which measure the force of mortality from lung cancer and other diseases due to substances Y and X contained in tobacco smoke) in the instantaneous risks β'_1 and β'_2 (which measure the force of mortality from natural causes). Let $\beta'_1 + \beta_1 = \beta_{11}$, $\beta'_2 + \beta_2 = \beta_{22}$, $\beta_{11} + \beta_{22} = \beta_T$. Then the crude probabilities among smokers, of death from lung cancer, other diseases, and all causes respectively, are

$$Q_{1t} = \frac{\beta_{11}}{\beta_T}\left[1 - e^{-\beta_T t}\right] \tag{51}$$

$$Q_{2t} = \frac{\beta_{22}}{\beta_T}[1 - e^{-\beta_T t}] \tag{52}$$

$$Q_t = 1 - e^{-\beta_T t} \tag{53}$$

N' nonsmokers have been observed, of whom d'_1 have died from lung cancer at times $t'_{11}, t'_{12}, \cdots, t'_{1d'_1}$, and d'_2 from other diseases at times $t'_{21}, t'_{22}, \cdots, t'_{2d'_2}$ and $s' = N' - d'_1 - d'_2$ have survived to the end of the unit period. N smokers have been observed of whom d_1 have died from lung cancer at times $t_{11}, t_{12}, \cdots, t_{1d_1}$, and d_2 from other diseases at times $t_{21}, t_{22}, \cdots, t_{2d_2}$, and $s = N - d_1 - d_2$ have survived to the end of the unit period. We wish to obtain the maximum likelihood estimates of $\beta'_1, \beta'_2, \beta_1, \beta_2$, the corresponding net probabilities q, and also the crude probabilities Q of death from lung cancer and from other diseases, attributable to natural causes and attributable to smoking.

The joint sample probability for the total sample of experience, the deaths from the several causes and their times among the nonsmokers, and also the corresponding observations among the smokers, can be set down, and the estimates and their variances derived in the standard way as illustrated in the previous sections. However, most of the estimates wanted may be had directly from those derived in previous sections, from the following considerations.

Using the observations on the nonsmokers, the time estimates of β'_1, β'_2 are given by (23), and the frequency estimates by (33), (34). Similarly, from the observations on the smokers the estimates of β_{11} and β_{22} are given by the same formulas.

Since the observations on the nonsmokers and the smokers are made on different samples the estimates $\hat{\beta}_{11}$ and $\hat{\beta}'_1$ are independent, as are those of $\hat{\beta}_{22}$ and $\hat{\beta}'_2$. Then

$$\hat{\beta}_1 = \hat{\beta}_{11} - \hat{\beta}'_1$$
$$\hat{\beta}_2 = \hat{\beta}_{22} - \hat{\beta}'_2$$

and

$$\text{var. } \hat{\beta}_1 = \text{var. } \hat{\beta}_{11} + \text{var. } \hat{\beta}'_1$$
$$\text{var. } \hat{\beta}_2 = \text{var. } \hat{\beta}_{22} + \text{var. } \hat{\beta}'_2.$$

As for the q's and Q's, these are functions of the β's, and their estimates are the same functions of the estimates of the β's. The variances of the $\hat{q}$'s and $\hat{Q}$'s are derivable from the variances and covariances of the $\hat{\beta}$'s.

The estimates and their variances for the chief parameters of interest are given in Table 2.

Example.—We take, as an example of the application of the derived formulas, some data from the prospective study sponsored by the American Cancer Society and reported by Hammond and Horn [10], [11], [12]. Some 200,000 men in the age range 50 to 70 years were interviewed and a statement obtained from each as to his smoking habits. Periodically, inquiry was made and it was ascertained when any individual had died, and the time and cause of death as stated on the death certificate were recorded. A report was made based on the

TABLE 2. MAXIMUM LIKELIHOOD ESTIMATES AND VARIANCES

θ	Time Estimates		Frequency Estimates		Refers to deaths from
	$\hat{\theta}$	$\sigma_{\hat{\theta}}^2$	$\hat{\theta}$	$\sigma_{\hat{\theta}}^2$	
			Instantaneous risks		
β'_1	$\dfrac{d'_1}{\Sigma t' + s'}$	$\dfrac{\beta'^2_1 + \beta'_1\beta'_2}{N'(1 - e^{-\beta'})}$	$\dfrac{d'_1}{d'} \ln \dfrac{N'}{s'}$	$\dfrac{1}{N'}\left[\dfrac{\beta'^2_1(1 - e^{-\beta})}{\beta'^2 e^{-\beta'}} + \dfrac{\beta'_1\beta'_2}{1 - e^{-\beta'}}\right]$	lung cancer, due to natural causes
β'_2	$\dfrac{d'_2}{\Sigma t' + s'}$	$\dfrac{\beta'^2_2 + \beta'_1\beta'_2}{N'(1 - e^{-\beta})}$	$\dfrac{d'_2}{d'} \ln \dfrac{N'}{s'}$	$\dfrac{1}{N'}\left[\dfrac{\beta'^2_2(1 - e^{-\beta})}{\beta'^2 e^{-\beta'}} + \dfrac{\beta'_1\beta'_2}{1 - e^{-\beta'}}\right]$	other diseases, due to natural causes
β'	$\dfrac{d'}{\Sigma t' + s'}$	$\dfrac{\beta'^2}{N(1 - e^{-\beta'})}$	$\ln \dfrac{N'}{s'}$	$\dfrac{1 - e^{-\beta}}{N'e^{-\beta}} = \dfrac{Q'}{N'P'}$	all diseases, due to natural causes
β_{11}	$\dfrac{d_1}{\Sigma t + s}$	$\dfrac{\beta^2_{11} + \beta_{11}\beta_{22}}{N(1 - e^{-\beta_T})}$	$\dfrac{d_1}{d} \ln \dfrac{N}{s}$	$\dfrac{1}{N}\left[\dfrac{\beta_{11}{}^2(1 - e^{-\beta_T})}{\beta_T{}^2 e^{-\beta_T}} + \dfrac{\beta_{11}\beta_{22}}{1 - e^{-\beta_T}}\right]$	lung cancer, due to natural causes plus smoking
β_{22}	$\dfrac{d_2}{\Sigma t + s}$	$\dfrac{\beta^2_{22} + \beta_{11}\beta_{22}}{N(1 - e^{-\beta_T})}$	$\dfrac{d_2}{d} \ln \dfrac{N}{s}$	$\dfrac{1}{N}\left[\dfrac{\beta_{22}{}^2(1 - e^{-\beta_T})}{\beta_T{}^2 e^{-\beta_T}} + \dfrac{\beta_{11}\beta_{22}}{1 - e^{-\beta_T}}\right]$	other diseases, due to natural causes plus smoking
β_T	$\dfrac{d}{\Sigma t + s}$	$\dfrac{\beta_T{}^2}{N(1 - e^{-\beta_T})}$	$\beta_{11} + \beta_{22} = \ln \dfrac{N}{s}$	$\dfrac{1 - e^{-\beta_T}}{Ne^{-\beta_T}} = \dfrac{Q}{NP}$	all diseases, due to natural causes plus smoking
β_1	$\hat{\beta}_{11} - \hat{\beta}'_1$	$\sigma_{\hat{\beta}_{11}}{}^2 + \sigma_{\hat{\beta}'_1}{}^2$	$\hat{\beta}_{11} - \hat{\beta}'_1$	$\sigma_{\hat{\beta}_{11}}{}^2 + \sigma_{\hat{\beta}'_1}{}^2$	lung cancer, due to smoking
β_2	$\hat{\beta}_{22} - \hat{\beta}'_2$	$\sigma_{\hat{\beta}_{22}}{}^2 + \sigma_{\hat{\beta}'_2}{}^2$	$\hat{\beta}_{22} - \hat{\beta}'_2$	$\sigma_{\hat{\beta}_{22}}{}^2 + \sigma_{\hat{\beta}'_2}{}^2$	other diseases, due to smoking
β	$\hat{\beta}_T - \hat{\beta}'$	$\sigma_{\hat{\beta}_T}{}^2 + \sigma_{\hat{\beta}'}{}^2$	$\hat{\beta}_T - \hat{\beta}' = \ln \dfrac{N}{N'} \dfrac{S'}{S}$	$\sigma_{\hat{\beta}T}{}^2 + \sigma_{\hat{\beta}'}{}^2 = \dfrac{Q}{NP} + \dfrac{Q'}{N'P'}$	all diseases, due to smoking

(*Continued on next page*)

TABLE 2—(*cont.*)

θ	Time Estimates		Frequency Estimates		Refers to deaths from
	$\hat{\theta}$	σ_θ^2	$\hat{\theta}$	σ_θ^2	
			Death rates		
q'_1	$1 - e^{-\hat{\beta}'_1}$	$\sigma_{\hat{\beta}'_1}{}^2 e^{-2\beta}$	$1 - e^{-\hat{\beta}'_1}$	$\sigma_{\hat{\beta}'_1}{}^2 e^{-2\beta'_1}$	lung cancer, due to natural causes
q'_2	$1 - e^{-\hat{\beta}'_2}$	$\sigma_{\hat{\beta}'_2}{}^2 e^{-2\beta}{}_2$	$1 - e^{-\hat{\beta}'_2}$	$\sigma_{\hat{\beta}'_2}{}^2 e^{-2\beta'_2}$	other diseases, due to natural causes
Q'	$1 - e^{-\hat{\beta}'}$	$\sigma_{\hat{\beta}'}{}^2 e^{-2\beta}$	$1 - e^{-\hat{\beta}'} = \dfrac{d'}{N'}$	$\sigma_{\hat{\beta}'}{}^2 e^{-2\beta} = \dfrac{P'Q'}{N'}$	all diseases, due to natural causes
q_1	$1 - e^{-\hat{\beta}_1}$	$\sigma_{\hat{\beta}_1}{}^2 e^{-2\beta_1}$	$1 - e^{-\hat{\beta}_1}$	$\sigma_{\hat{\beta}_1}{}^2 e^{-2\beta_1}$	lung cancer, due to smoking
q_2	$1 - e^{-\hat{\beta}_2}$	$\sigma_{\hat{\beta}_2}{}^2 e^{-2\beta_2}$	$1 - e^{-\hat{\beta}_2}$	$\sigma_{\hat{\beta}_2}{}^2 e^{-2\beta_2}$	other diseases, due to smoking
q	$1 - e^{-\beta}$	$\sigma_{\hat{\beta}}{}^2 e^{-2\beta}$	$1 - e^{-\hat{\beta}} = \dfrac{(d/N) - (d'/N')}{1 - (d'/N')}$*	$\sigma_{\hat{\beta}}{}^2 e^{-2\beta} = \left(\dfrac{Q}{NP} + \dfrac{Q'}{N'P'}\right)\left(\dfrac{P}{P'}\right)^2$*	all diseases, due to smoking
Q	$1 - e^{-\hat{\beta}_T}$	$\sigma_{\hat{\beta}}{}^2{}_T e^{-2\beta T}$	$1 - e^{-\hat{\beta}_T} = \dfrac{d}{N}$	$\sigma_{\hat{\beta}T}{}^2 e^{-2\beta T} = \dfrac{PQ}{N}$	all diseases, due to natural causes plus smoking

* These formulas are given by Sheps [16], in a different notation.

TABLE 3. DEATHS IN 44 MONTHS. AGE AT BEGINNING, 60–64 YEARS*

Cause of death	Nonsmokers n = 20,278		Smokers n = 21,594		Measure of effect of smoking		
	Deaths = d		Deaths = d		Net prob. due to smoking†	Diff. of prob.†	Mort. ratio
	No.	d/n Prob.†	No.	d/n Prob.†			
Lung cancer	10	.00049	103	.00477	.00449	.00428	9.7
Other cancer	218	.01075	325	.01505	.00469	.00430	1.4
Coronary disease	552	.02722	921	.04265	.01653	.01543	1.6
Other diseases	486	.02397	667	.03089	.00765	.00692	1.3
Total: All causes	1,266	.06243	2,016	.09336	.03299	.03093	1.5

* Data and categories of Hammond and Horn; nonsmokers = never smoked cigarettes regularly; smokers = all regular cigarette smokers.

† These are, of course, estimates. Their standard errors are, with such large numbers as are involved in these investigations, small; the net probability due to smoking is "statistically significant," at level $P < .001$ for each of the categories of death.

status of each individual as of 44 months after the initial inquiry. In Table 3 are shown the essential data for the group of men 60–65 years of age at the time of the original inquiry. The binomial estimates of the probability of death in the 44-month period of follow-up are shown for the nonsmokers and for the smokers, and for each of four categories of cause of death, namely cancer of the lung, other cancer, coronary-artery disease, and other diseases, as well as for deaths from all diseases. In the last three columns are shown three indices of the effect of smoking in increasing the probability of death from each of the categorized causes. The first of these indices is the estimate of the net probability of death from the respective causes, using the frequency maximum likelihood estimate. The second is the simple difference of the probabilities of death of smokers and of nonsmokers. The third column gives the "mortality ratio," which here is the ratio of the probability of death among smokers to the probability among nonsmokers.

If we use the net probability of death as the measure of the effect of smoking in respect of a cause of death, we see that, among the four categorized causes of death, smoking has the greatest effect in increasing deaths from coronary heart disease, the next greatest from diseases in the class "other diseases," the next from cancer other than lung cancer, and the least from lung cancer. If we use the simple difference of the probabilities of death, shown in the next column, we reach essentially the same conclusion. This is not surprising, since it is easily shown that if the probabilities of death Q, q are small, the net probabilities are given with close approximation by the simple difference of the probabilities among smokers and nonsmokers. On the basis of either of these estimates, the effect of smoking in causing death from lung cancer is only about 14 per cent of the effect in causing death from all diseases, that is, the effect of bringing on death from diseases other than lung cancer is more than six times the effect of causing death from lung cancer.

If we use the "mortality ratio" shown in the last column, a quite different impression is obtained. The impression then is that for each of the categories other than lung cancer, the effect of smoking is about the same, and, compared with the effect on lung cancer, is negligible. We see that the mortality ratio is 9.7 for lung cancer, while it is less than 2 for each of the other categories. In one important report of a prospective study, [3] a ratio of less than 2 was considered as hardly worth commenting on as physically significant. If this view is applied to the statistics of the table presented, it would be said only that these data are evidence that smoking causes lung cancer. Indeed, this is the way these and similar findings in other prospective studies have been interpreted. In a number of published articles dealing with the effect of smoking, which depend largely on these prospective studies, only lung cancer is mentioned! The finding of associations of smoking with classes of disease in general, not only, or even chiefly, with lung cancer, is of central importance for an interpretation of the possible biologic significance of these statistical studies [2], [13].

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A NONPARAMETRIC SUM OF RANKS PROCEDURE FOR RELATIVE SPREAD IN UNPAIRED SAMPLES[1]

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A nonparametric procedure is presented to test the null hypothesis that two independent samples come from the same population against the alternative hypothesis that the samples come from populations differing in variability or "spread." Extensive tables of critical values are included for $n_1 \leq n_2 \leq 20$. Large sample procedures are presented which include a correction for tied observations. The test is entirely distribution-free under the usual randomization procedures against the null hypothesis that the two distributions are identical. The absence of any normality assumption is a particularly important feature of the test, because its parametric alternative, the F test for variance differences, is quite sensitive to departures from normality. The test has the additional advantage of being directly applicable to non-numerical ordinal data.

THE procedure developed and discussed in this paper offers a nonparametric test of the null hypothesis that two independent samples come from the same population, a test which has considerable power against the alternative hypothesis that the samples come from populations differing in variability or "spread" but which is relatively insensitive to alternative hypotheses that the two populations differ mainly in location. The test is sensitive to differences in variability when the location parameters, e.g. means, of the populations are equal or nearly equal.

Tables of critical values for samples of sizes ≤ 20 and single-tail probabilities of 0%, 0.1%, 0.5%, 1%, 2.5%, 5% and 10% are given. The two sample sizes may be equal or different. These tables are equally applicable to the Wilcoxon test [18, 20]. Convenient approximations for cases falling outside the scope of the tables are discussed.

The absence of any normality assumption is a particularly important feature of this test because its parametric alternative, the F test for differences in variances, is quite sensitive to departures from normality. This was first pointed out in 1931 by E. S. Pearson [16], and his findings have been confirmed by Geary [10], Finch [6], and Gayen [9]. Box [3] has shown that the sensitivity to departures from normality may be even greater when more than two variances are compared by Bartlett's homogeneity of variance test.

A number of attempts have been made recently, notably by Lehmann [13], Mood [15], and Sukhatme [19], to develop nonparametric tests of variance differences. The procedure suggested by Lehmann, however, has been shown (by Mood and by Sukhatme) not to be distribution free. The test proposed by Mood assumes knowledge of the means of the two populations, whereas in practice such knowledge rarely exists. The procedure suggested by

[1] This paper reports work done when the authors were Fellows at the Center for Advanced Study in the Behavioral Sciences at Stanford, California, 1957–58.

Sukhatme also assumes knowledge regarding the location of the two populations, and in addition this procedure is not as efficient as the one proposed by Mood. Procedures proposed by Kamat [11] and Rosenbaum [17] seem likely to lack power. Ansari and Bradley [1] have recently developed a sum of ranks test for dispersion. Additional work on this topic has been presented by Freund and Ansari [8] and by David and Barton [5].

To illustrate the test procedure we propose, we may consider data from a study discussed elsewhere ([18], pp. 148–51) in which the amount of aggression attributed to film characters by members of two populations, A and B, was compared.

Sample from A:	25	5	14	19	0	17	15	8	8
Sample from B:	12	16	6	13	13	3	10	10	11

In the present procedure, the observations are first combined in a single series, in order of increasing (or decreasing) value, retaining their identification as *a*'s or *b*'s:

Score:	0	3	5	6	8	8	10	10	11	12	13	13	14	15	16	17	19	25
Sample:	*a*	*b*	*a*	*b*	*a*	*a*	*b*	*b*	*b*	*b*	*b*	*b*	*a*	*a*	*b*	*a*	*a*	*a*

(When ties occur between two observations from the same sample, as in the above series, the order in which we arrange the observation does not matter. When observations are tied across samples, more caution is required. The whole question of tied observations is discussed in a later section of this paper.)

If we were interested in testing the null hypothesis that these samples are from the same population or from populations with identical distribution functions against the alternative that the populations differ in *location*, we might assign ranks 1 to 18 to this ordering in a direct manner, from lowest to highest, determine the sum of the ranks of the *a*'s (or *b*'s), and then proceed with a rank sum test of a Wilcoxon type [20], one version of which is the Mann-Whitney *U* test [14].

However, since we are interested in testing the null hypothesis against the alternative that the populations differ in *variability*, we shall assign ranks 1 to 18 to the ordered observations in a different manner, one which assigns low ranks to extreme observations and high ranks to central observations. We assign rank 1 to the lowest number of the sequence, ranks 2 and 3 to the two highest members in the sequence, ranks 4 and 5 to the next two lowest, etc. (If the total number of observations is odd, the middlemost score is dropped—not assigned a rank—in order that the highest assigned rank will be even.) This ranking procedure is illustrated with the data already presented:

Score:	0	3	5	6	8	8	10	10	11	12	13	13	14	15	16	17	19	25
Sample:	*a*	*b*	*a*	*b*	*a*	*a*	*b*	*b*	*b*	*b*	*b*	*b*	*a*	*a*	*b*	*a*	*a*	*a*
Rank:	1	4	5	8	9	12	13	16	17	18	15	14	11	10	7	6	3	2

Assigning the ranks in this way puts the lower ranks at the extremes of the ordered sequence and the higher ranks in the middle of the sequence. If the null hypothesis were true, the observations from the two populations would tend to be well mixed, so that the mean rank assigned to one of the samples would tend to equal the mean rank assigned to the other sample. If, on the

other hand, the alternative hypothesis were true, we would expect more of the observations from the population with greater spread to be near the extremes of the ordered sequence and, therefore, to be assigned the lower ranks, and we would expect more of the observations from the less variable population to be near the middle of the sequence, and, therefore, to be assigned the higher ranks. Thus we would expect the mean rank assigned to the observations from the more variable population to be considerably smaller than the mean rank assigned to the observations from the less variable population.

Since the total sum of ranks is fixed, we may work with the sum of ranks for either group. If the two groups are of different size, arithmetic convenience causes us to choose the sum of ranks for the smaller group. In the example presented above, the sum of ranks for the a's is $R_a = 59$. (Had we used the same ranking procedure but started with 1 for the highest value, 2 and 3 for the lowest two values, etc., we would have found $R_a' = 60$.)

One reason for choosing the above ranking procedure is that it allows the use of the same tables as are used for the Wilcoxon rank sum test for location. Thus our procedure may be considered a Wilcoxon test for spread in unpaired samples.

The standard tables apply directly and precisely to either way of ranking the observations so long as it is chosen in advance of seeing the observations. As a consequence, when the standard tables are applied to the mean of the two rank sums, $\frac{1}{2}(R_a + R_a')$, the result may be shown to be slightly conservative.

PROCEDURE WHEN NEITHER SAMPLE CONTAINS MORE THAN TWENTY OBSERVATIONS

Let $n_1 \leq n_2$ be the sizes of the two samples. Call the sample of size n_1 the first sample. Let R_1 be the sum of ranks for the first sample. Table 1 gives critical values of R_1 for $n_1 \leq n_2 \leq 20$ and single tail probabilities 0%, 0.1%, 0.5%, 1%, 2.5%, 5%, and 10%.[2] As was noted above, these tables are equally applicable to the ordinary (Wilcoxon, Mann-Whitney, White, Festinger) rank sum procedure for relative location of two unpaired samples. The null-hypothesis probability of obtaining a smaller-sample rank sum *less than or equal to* any entry in the *left half* of the table is not greater than the smaller percentage at the head of the column containing the entry. The null-hypothesis probability of obtaining a smaller-sample rank sum *greater than or equal to* any entry in the *right half* of the table is not greater than the smaller percentage at the head of the column containing the entry. Where the least extreme rank sum which is significant at a given tabular level is also significant at a more extreme level, this fact is indicated by the entry of an arrow. (The parenthetic values for 0% and 100% are, as these statements imply, those values which are just barely impossible.) The tabular values are appropriate linear transformations of the values presented by Auble [2].

In the example discussed earlier, $n_1 = n_2 = 9$, and $R_1 = 59$. Reference to the appropriate table shows this value to be significantly low at a one-tailed 1% level, but not at a one-tailed 0.5% level. Since low values correspond to a

[2] The proposed test is not recommended for $n_1 = 1$. The tables include values for $n_1 = 1$ for purposes of completeness.

greater spread for sample A, we can conclude with 99% confidence that population A is more spread out than population B.

PROCEDURE WITH TWO LARGE SAMPLES

Taking the mean and variance of R_1, from the Appendix, and making a continuity correction, it is appropriate to treat

$$z = \frac{2R_1 - n_1(n_1 + n_2 + 1) \pm 1}{\sqrt{n_1(n_1 + n_2 + 1)(n_2/3)}} \tag{1}$$

as a unit approximately normal deviate (under the null hypothesis), provided we choose that sign in the numerator which makes the magnitude of z smaller.

In the example, where the sample sizes are not really large, this procedure yields

$$z = \frac{2(59) - 171 + 1}{\sqrt{513}}$$

$$= -2.290$$

which falls at the one-sided 1.1% point, in reasonable agreement with our previous result.

PROCEDURE WITH BADLY UNBALANCED SAMPLES, AT LEAST ONE LARGE

If one sample is small and the other is large, the quantity z of (1) above may not be sufficiently approximately normally distributed. If such is the case we may find it desirable to adjust z so as to improve the adequacy of the approximation.

Two of the possible adjustments are discussed in the Appendix. The recommended choice is

$$z_s = z + \left(\frac{1}{10n_1} - \frac{1}{10n_2}\right)(z^3 - 3z) \tag{2}$$

which is shown to perform quite well for rather extreme conditions.

Outside the range of the present tables, z will usually be adequate, but z_s may be easily calculated in cases of doubt. When $n_1 = 2$ and n_2 is large, direct calculation of the probability may be advisable, using , with any integer j_1,

$$\text{Prob}\,(R_1 \leq 2j) = \frac{2j(j-1)}{(n_2+1)(n_2+2)} \qquad \text{(when } n_1 = 2\text{)}$$

$$\text{Prob}\,(R_1 \leq 2j+1) = \frac{2j^2}{(n_2+1)(n_2+2)}$$

TREATMENT OF TIES

If two tied observations come from the same sample, treating them as arranged in either order does not affect the values of R_1. It is thus possible to neglect such ties in calculating R_1.

If two observations from different samples are tied, "breaking" the tie in the two possible ways would lead to different values of R_1. It is, therefore, probable that we should, in such cases, wish to assign average ranks to the tied observations. If we do this, we shall presumably have to allow for its effects in assessing significance.

A rather similar situation arises if there are no ties but we wish to average the two sums of ranks obtained by starting, respectively, with the lowest and highest observations. We shall then assign to two observations the rank 1.5, to two others the rank 3.5 and so on.

If both samples are at least moderately large, in any of these cases we need only correct the variance of R_1 and continue to use the modified z which results. This change in variance depends, as we show in the Appendix, only on the reduction in sums of squares of ranks assigned, and may be compensated for by using as the denominator of z in (1)

$$\sqrt{n_1(n_1 + n_2 + 1)(n_2/3) - 4[n_1n_2/(n_1 + n_2)(n_1 + n_2 - 1)](S_1 - S_2)} \qquad (3)$$

where S_1 is the sum of squares of the ranks (not averaged) of the tied observations, and S_2 is the sum of squares of the averaged ranks of the tied observations.

The reduction in sum of squares may be simply and directly computed. Referring again to the example, where tied pairs of observations were ranked 9 and 12, 13 and 16, and 14 and 15 respectively, if we were to replace these by the corresponding average ranks—10.5 and 10.5, 14.5 and 14.5, and again 14.5 and 14.5, the contribution of these values to the sum of squares would change from

$$S_1 = 9^2 + 12^2 + 13^2 + 16^2 + 14^2 + 15^2 = 1071.0$$

to

$$S_2 = 10.5^2 + 10.5^2 + 14.5^2 + 14.5^2 + 14.5^2 + 14.5^2 = 1061.5$$

the reduction thus being 9.5, which would alter the denominator of z from $\sqrt{513}$ to $\sqrt{502.94}$ and would thus increase the magnitude of z from -2.290 to -2.319.

While this particular change is not of any great magnitude, we clearly ought not to have a choice as to whether or not we accept it. If we are going to use average ranks for ties coming from different samples, and if we wish to maintain the permutation nature of our procedure, we *should* use average ranks for all ties. Since using average ranks for ties within a single sample does not affect R_1, since its effect on z is always to make it more significant, we may, if we wish, neglect to allow for the use of average ranks within samples, but our significance statements will then be more conservative than necessary.

DISCUSSION

The application of the test we have described does not require any assumption that the distribution(s) from which the samples were drawn is normal or any other specific shape. Against the null hypothesis that the two distributions are identical, the test is entirely distribution-free under the usual randomization procedures.

The test has the additional advantage of being directly applicable to non-numerical ordinal data. Observations which are ordered but not measured on any scale arise frequently, especially in research in the behavioral sciences.

It should be noted that in the presence of differences between population means the test tends to lose its sensitivity to differences in variability. The larger the difference in means, the lower the probability that the null hypothesis will be rejected even in the presence of differences in variability. That is, the larger the difference in means, the larger the probability of a Type II Error. This is especially true, of course, when the variabilities are small. In the case where the populations are disjoint (have no overlap), the power of the test is zero.

It should also be noted, however, that differences between means, in every instance so far examined, *decrease* the probability of a Type I Error. In general, rejection of the null hypothesis can almost certainly be attributed to differences in variability.

The power-efficiency of the test against normal alternatives has not been investigated as yet. Inasmuch as the two-sample F text for variance differences loses both efficiency and validity rapidly in the presence of non-normality, the test proposed here should serve as a useful and safer alternative when little is known about the population distributions.

TABLE 1. TABLES OF EXACT CRITICAL VALUES FOR R_1

R_1 = sum of ranks for smaller sample, n_1 = size of smaller sample, n_2 = size of larger sample. One-tail % given above blocks; two-tail % given below blocks. (Also applicable to the Wilcoxon test [18, 20] in its original form.)

$n_1 = 1$

n_2	100 / 0	99.9 / 0.1	99.5 / 0.5	99 / 1	97.5 / 2.5	95 / 5	90 / 10	%→ ←%	10 / 90	5 / 95	2.5 / 97.5	1 / 99	0.5 / 99.5	0.1 / 99.9	0 / 100
≤8	(R_1 never reaches a one-sided 10% level of significance for $n_2 \leq 8$)														
9	0						1		10						11
10	0						1		11						12
11	0						1		12						13
12	0						1		13						14
13	0						1		14						15
14	0						1		15						16
15	0						1		16						17
16	0						1		17						18
17	0					—	1		18	—					19
18	0					—	1		19	—					20
19	0	—	—	—	—	1	2		19	20	—	—	—	—	21
20	0	—	—	—	—	1	2		20	20	—	—	—	—	22
%	0	0.2	1	2	5	10	20	←→	20	10	5	2	1	0.2	0

TABLE 1—(*cont.*)

$n_1 = 2$

n_2	100 0	99.9 0.1	99.5 0.5	99 1	97.5 2.5	95 5	90 10	%→ ←%	10 90	5 95	2.5 97.5	1 99	0.5 99.5	0.1 99.9	0 100
2	2						—		—						8
3	2					—	3		9	—					10
4	2					—	3		11	—					12
5	2					3	4		12	13					14
6	2				—	3	4		14	15	—				16
7	2				—	3	4		16	17	—				18
8	2				3	4	5		17	18	19				20
9	2				3	4	5		19	20	21				22
10	2				3	4	6		20	22	23				24
11	2			—	3	4	6		22	24	25	—			26
12	2			—	4	5	7		23	25	26	—			28
13	2			3	4	5	7		25	27	28	29			30
14	2			3	4	5	7		27	29	30	31			32
15	2			3	4	6	8		28	30	32	33			34
16	2			3	4	6	8		30	32	34	35			36
17	2		—	3	5	6	9		31	34	35	37	—		38
18	2		—	3	5	7	9		33	35	37	39	—		40
19	2	—	3	4	5	7	10		34	37	39	40	41	—	42
20	2	—	3	4	5	7	10		36	39	41	42	43	—	44
%	0	0.2	1	2	5	10	20	⟷	20	10	5	2	1	0.2	0

$n_1 = 3$

n_2	100 1	99.9 0.1	99.5 0.5	99 1	97.5 2.5	95 5	90 10	%→ ←%	10 90	5 95	2.5 97.5	1 99	0.5 99.5	0.1 99.9	0 100
3	5				—	6	7		14	15	—				16
4	5				—	6	7		17	18	—				19
5	5			—	6	7	8		19	20	21	—			22
6	5			—	7	8	9		21	22	23	—			25
7	5		—	6	7	8	10		23	25	26	27	—		28
8	5		—	6	8	9	11		25	27	28	30	—		31
9	5		6	7	8	9	11		28	30	31	32	33		34
10	5		6	7	9	10	12		30	32	33	35	36		37
11	5		6	7	9	11	13		32	34	36	38	39		40
12	5		7	8	10	11	14		34	37	38	40	41		43
13	5		7	8	10	12	15		36	39	41	43	44		46
14	5		7	8	11	13	16		38	41	43	46	47		49
15	5	—	8	9	11	13	16		41	44	46	48	49	—	52
16	5	—	8	9	12	14	17		43	46	48	51	52	—	55
17	5	6	8	10	12	15	18		45	48	51	53	55	57	58
18	5	6	8	10	13	15	19		47	51	53	56	58	60	61
19	5	6	9	10	13	16	20		49	53	56	59	60	63	64
20	5	6	9	11	14	17	21		51	55	58	61	63	66	67
%	0	0.2	1	2	5	10	20	⟷	20	10	5	2	1	0.2	0

TABLE 1—(*cont.*)

$n_1 = 4$

n_2	100 1	99.9 0.1	99.5 0.5	99 1	97.5 2.5	95 5	90 10	%→ ←%	10 90	5 95	2.5 97.5	1 99	0.5 99.5	0.1 99.9	0 100
4	9		—	—	10	11	13		23	25	26	—	—		27
5	9		—	10	11	12	14		26	28	29	30	—		31
6	9		10	11	12	13	15		29	31	32	33	34		35
7	9		10	11	13	14	16		32	34	35	37	38		39
8	9	—	11	12	14	15	17		35	37	38	40	41	—	43
9	9	—	11	13	14	16	19		37	40	42	43	45	—	47
10	9	10	12	13	15	17	20		40	43	45	47	48	50	51
11	9	10	12	14	16	18	21		43	46	48	50	52	54	55
12	9	10	13	15	17	19	22		46	49	51	53	55	58	59
13	9	11	13	15	18	20	23		49	52	54	57	59	61	63
14	9	11	14	16	19	21	25		51	55	57	60	62	65	67
15	9	11	15	17	20	22	26		54	58	60	63	65	69	71
16	9	12	15	17	21	24	27		57	60	63	67	69	72	75
17	9	12	16	18	21	25	28		60	63	67	70	72	76	79
18	9	13	16	19	22	26	30		62	66	70	73	76	79	83
19	9	13	17	19	23	27	31		65	69	73	77	79	83	87
20	9	13	18	20	23	28	32		68	72	77	80	82	87	91
%	0	0.2	1	2	5	10	20	⟷	20	10	5	2	1	0.2	0

$n_1 = 5$

n_2	100 0	99.9 0.1	99.5 0.5	99 1	97.5 2.5	95 5	90 10	%→ ←%	10 90	5 95	2.5 97.5	1 99	0.5 99.5	0.1 99.9	0 100
5	14		15	16	17	19	20		35	36	38	39	40		41
6	14	—	16	17	18	20	22		38	40	42	43	44	—	46
7	14	—	16	18	20	21	23		42	44	45	47	49	—	51
8	14	15	17	19	21	23	25		45	47	49	51	53	55	56
9	14	16	18	20	22	24	27		48	51	53	55	57	59	61
10	14	16	19	21	23	26	28		52	54	57	59	61	64	66
11	14	17	20	22	24	27	30		55	58	61	63	65	68	71
12	14	17	21	23	26	28	32		58	62	64	67	69	73	76
13	14	18	22	24	27	30	33		62	65	68	71	73	77	81
14	14	18	22	25	28	31	35		65	69	72	75	78	82	86
15	14	19	23	26	29	33	37		68	72	76	79	82	86	91
16	14	20	24	27	30	34	38		72	76	80	83	86	90	96
17	14	20	25	28	32	35	40		75	80	83	87	90	95	101
18	14	21	26	29	33	37	42		78	83	87	91	94	99	106
19	14	22	27	30	34	38	43		82	87	91	95	98	103	111
20	14	22	28	31	35	40	45		85	90	95	99	102	108	116
%	0	0.2	1	2	5	10	20	⟷	20	10	5	2	1	0.2	0

TABLE 1—(*cont.*)

$n_1 = 6$

n_2	100 / 1	99.9 / 0.1	99.5 / 0.5	99 / 1	97.5 / 2.5	95 / 5	90 / 10	%→ ←%	10 / 90	5 / 95	2.5 / 97.5	1 / 99	0.5 / 99.5	0.1 / 99.9	0 / 100
6	20	—	23	24	26	28	30		48	50	52	54	55	—	58
7	20	21	24	25	27	29	32		52	55	57	59	60	63	64
8	20	22	25	27	29	31	34		56	59	61	63	65	68	70
9	20	23	26	28	31	33	36		60	63	65	68	70	73	76
10	20	24	27	29	32	35	38		64	67	70	73	75	78	82
11	20	25	28	30	34	37	40		68	71	74	78	80	83	88
12	20	25	30	32	35	38	42		72	76	79	82	84	89	94
13	20	26	31	33	37	40	44		76	80	83	87	89	95	100
14	20	27	32	34	38	42	46		80	84	88	92	94	99	106
15	20	28	33	36	40	44	48		84	88	92	96	99	104	112
16	20	29	34	37	42	46	50		88	92	96	101	104	109	118
17	20	30	36	39	43	47	52		92	97	101	105	108	114	124
18	20	31	37	40	45	49	55		95	101	105	110	113	119	130
19	20	32	38	41	46	51	57		99	105	110	115	118	124	136
20	20	33	39	43	48	53	59		103	109	114	119	123	129	142
%	0	0.2	1	2	5	10	20	←→	20	10	5	2	1	0.2	0

$n_1 = 7$

n_2	100 / 0	99.9 / 0.1	99.5 / 0.5	99 / 1	97.5 / 2.5	95 / 5	90 / 10	%→ ←%	10 / 90	5 / 95	2.5 / 97.5	1 / 99	0.5 / 99.5	0.1 / 99.9	0 / 100
7	27	29	32	34	36	39	41		64	66	69	71	73	76	78
8	27	30	34	35	38	41	44		68	71	74	77	78	82	85
9	27	31	35	37	40	43	46		73	76	79	82	84	88	92
10	27	33	37	39	42	45	49		77	81	84	87	89	93	99
11	27	34	38	40	44	47	51		82	86	89	93	95	99	106
12	27	35	40	42	46	49	54		86	91	94	98	100	105	113
13	27	36	41	44	48	52	56		91	95	99	103	106	111	120
14	27	37	43	45	50	54	59		95	100	104	109	111	117	127
15	27	38	44	47	52	56	61		100	105	109	114	117	123	134
16	27	39	46	49	54	58	64		104	110	114	119	122	129	141
17	27	41	47	51	56	61	66		109	114	119	124	128	134	148
18	27	42	49	52	58	63	69		113	119	124	130	133	140	155
19	27	43	50	54	60	65	71		118	124	129	135	139	146	162
20	27	44	52	56	62	67	74		122	129	134	140	144	152	169
%	0	0.2	1	2	5	10	20	←→	20	10	5	2	1	0.2	0

$n_1 = 8$

n_2	100 / 0	99.9 / 0.1	99.5 / 0.5	99 / 1	97.5 / 2.5	95 / 5	90 / 10	%→ ←%	10 / 90	5 / 95	2.5 / 97.5	1 / 99	0.5 / 99.5	0.1 / 99.9	0 / 100
8	35	40	43	45	49	51	55		81	85	87	91	93	96	101
9	35	41	45	47	51	54	58		86	90	93	97	99	103	109
10	35	42	47	49	53	56	60		92	96	99	103	105	110	117
11	35	44	49	51	55	59	63		97	101	105	109	111	116	125
12	35	45	51	53	58	62	66		102	106	110	115	117	123	133
13	35	47	53	56	60	64	69		107	112	116	120	123	129	141
14	35	48	54	58	62	67	72		112	117	122	126	130	136	149
15	35	50	56	60	65	69	75		117	123	127	132	136	142	157
16	35	51	58	62	67	72	78		122	128	133	138	142	149	165
17	35	53	60	64	70	75	81		127	133	138	144	148	155	173
18	35	54	62	66	72	77	84		132	139	144	150	154	162	181
19	35	56	64	68	74	80	87		137	144	150	156	160	168	189
20	35	57	66	70	77	83	90		142	149	155	162	166	175	197
%	0	0.2	1	2	5	10	20	←→	20	10	5	2	1	0.2	0

TABLE 1—(*cont.*)

$n_1 = 9$

n_2	100 0	99.9 0.1	99.5 0.5	99 1	97.5 2.5	95 5	90 10	%→ ←%	10 90	5 95	2.5 97.5	1 99	0.5 99.5	0.1 99.9	0 100
9	44	52	56	59	62	66	70		101	105	109	112	115	119	127
10	44	53	58	61	65	69	73		107	111	115	119	122	127	136
11	44	55	61	63	68	72	76		113	117	121	126	128	134	145
12	44	57	63	66	71	75	80		118	123	127	132	135	141	154
13	44	59	65	68	73	78	83		124	129	134	139	142	148	163
14	44	60	67	71	76	81	86		130	135	140	145	149	156	172
15	44	62	69	73	79	84	90		135	141	146	152	156	163	181
16	44	64	72	76	82	87	93		141	147	152	158	162	170	190
17	44	66	74	78	84	90	97		146	153	159	165	169	177	199
18	44	68	76	81	87	93	100		152	159	165	171	176	184	208
19	44	70	78	83	90	96	103		158	165	171	178	183	191	217
20	44	71	81	85	93	99	107		163	171	177	185	189	199	226
%	0	0.2	1	2	5	10	20	←→	20	10	5	2	1	0.2	0

$n_1 = 10$

n_2	100 0	99.9 0.1	99.5 0.5	99 1	97.5 2.5	95 5	90 10	%→ ←%	10 90	5 95	2.5 97.5	1 99	0.5 99.5	0.1 99.9	0 100
10	54	65	71	74	78	82	87		123	128	132	136	139	145	156
11	54	67	73	77	81	86	91		129	134	139	143	147	153	166
12	54	69	76	79	84	89	94		136	141	146	151	154	161	176
13	54	72	79	82	88	92	98		142	148	152	158	161	168	186
14	54	74	81	85	91	96	102		148	154	159	165	169	176	196
15	54	76	84	88	94	99	106		154	161	166	172	176	184	206
16	54	78	86	91	97	103	109		161	167	173	179	184	192	216
17	54	80	89	93	100	106	113		167	174	180	187	191	200	226
18	54	82	92	96	103	110	117		173	178	187	194	198	208	236
19	54	84	94	99	107	113	121		179	187	193	201	206	216	246
20	54	87	97	102	110	117	125		185	193	200	208	213	223	256
%	0	0.2	1	2	5	10	20	←→	20	10	5	2	1	0.2	0

$n_1 = 11$

n_2	100 0	99.9 0.1	99.5 0.5	99 1	97.5 2.5	95 5	90 10	%→ ←%	10 90	5 95	2.5 97.5	1 99	0.5 99.5	0.1 99.9	0 100
11	65	81	87	91	96	100	106		147	153	157	162	166	172	188
12	65	83	90	94	99	104	110		154	160	165	170	174	181	199
13	65	86	93	97	103	108	114		161	167	172	178	182	189	210
14	65	88	96	100	106	112	118		168	174	180	186	190	198	221
15	65	90	99	103	110	116	123		174	181	187	194	198	207	232
16	65	93	102	107	113	120	127		181	188	195	201	206	215	243
17	65	95	105	110	117	123	131		188	196	202	209	214	224	254
18	65	98	108	113	121	127	135		195	203	209	217	222	232	265
19	65	100	111	116	124	131	139		202	210	217	225	230	241	276
20	65	103	114	119	128	135	144		208	217	224	233	238	249	287
%	0	0.2	1	2	5	10	20	←→	20	10	5	2	1	0.2	0

TABLE 1—(*cont.*)

$n_1 = 12$

n_2	100 0	99.9 0.1	99.5 0.5	99 1	97.5 2.5	95 5	90 10	%→ ←%	10 90	5 95	2.5 97.5	1 99	0.5 99.5	0.1 99.9	0 100
12	77	98	105	109	115	120	127		173	180	185	191	195	202	223
13	77	101	109	113	119	125	131		181	187	193	199	203	211	235
14	77	103	112	116	123	129	136		188	195	201	208	212	221	247
15	77	106	115	120	127	133	141		195	203	209	216	221	230	259
16	77	109	119	124	131	138	145		203	210	217	224	229	239	271
17	77	112	122	127	135	142	150		210	218	225	233	238	248	283
18	77	115	125	131	139	146	155		217	226	233	241	247	257	295
19	77	118	129	134	143	150	159		225	234	241	250	255	266	307
20	77	120	132	138	147	155	164		232	241	249	258	264	276	319
%	0	0.2	1	2	5	10	20	⟷	20	10	5	2	1	0.2	0

$n_1 = 13$

n_2	100 0	99.9 0.1	99.5 0.5	99 1	97.5 2.5	95 5	90 10	%→ ←%	10 90	5 95	2.5 97.5	1 99	0.5 99.5	0.1 99.9	0 100
13	90	117	125	130	136	142	149		202	209	215	221	226	234	261
14	90	120	129	134	141	147	154		210	217	223	230	235	244	274
15	90	123	133	138	145	152	159		218	225	232	239	244	254	287
16	90	126	136	142	150	156	165		225	234	240	248	254	264	300
17	90	129	140	146	154	161	170		233	242	249	257	263	274	313
18	90	130	144	150	158	166	175		241	250	258	266	272	283	326
19	90	136	147	154	163	171	180		249	258	266	275	282	293	339
20	90	139	151	158	167	175	185		257	267	275	284	291	303	352
%	0	0.2	1	2	5	10	20	⟷	20	10	5	2	1	0.2	0

$n_1 = 14$

n_2	100 0	99.9 0.1	99.5 0.5	99 1	97.5 2.5	95 5	90 10	%→ ←%	10 90	5 95	2.5 97.5	1 99	0.5 99.5	0.1 99.9	0 100
14	104	137	147	152	160	166	174		232	240	246	254	259	269	302
15	104	141	151	156	164	171	179		241	249	256	264	269	279	316
16	104	144	155	161	169	176	185		249	258	265	273	279	290	330
17	104	148	159	165	172	182	190		258	266	276	283	289	300	344
18	104	151	163	170	179	187	196		266	275	283	292	299	311	358
19	104	155	168	174	183	192	202		274	284	293	302	308	321	372
20	104	159	172	178	188	197	207		283	293	302	312	318	331	386
%	0	0.2	1	2	5	10	20	⟷	20	10	5	2	1	0.2	0

$n_1 = 15$

n_2	100 0	99.9 0.1	99.5 0.5	99 1	97.5 2.5	95 5	90 10	%→ ←%	10 90	5 95	2.5 97.5	1 99	0.5 99.5	0.1 99.9	0 100
15	119	160	171	176	184	192	200		265	273	281	289	294	305	346
16	119	163	175	181	190	197	206		274	283	290	299	305	317	361
17	119	167	180	186	195	203	212		283	292	300	309	315	328	376
18	119	171	184	190	200	206	218		292	302	310	320	326	339	391
19	119	175	189	195	205	214	224		301	311	320	330	336	350	406
20	119	179	193	200	210	220	230		310	320	330	340	347	361	421
%	0	0.2	1	2	5	10	20	⟷	20	10	5	2	1	0.2	0

TABLE 1—(*cont.*)

$n_1 = 16$

n_2	100 0	99.9 0.1	99.5 0.5	99 1	97.5 2.5	95 5	90 10	%→ ←%	10 90	5 95	2.5 97.5	1 99	0.5 99.5	0.1 99.9	0 100
16	135	184	196	202	211	219	229		299	309	317	326	332	344	393
17	135	188	201	207	217	225	235		309	319	327	337	343	356	409
18	135	192	206	212	222	231	242		318	329	338	348	354	368	425
19	135	196	210	218	228	237	248		328	339	348	358	366	380	441
20	135	201	215	223	234	243	255		337	349	358	369	377	391	457
%	0	0.2	1	2	5	10	20	⟷	20	10	5	2	1	0.2	0

$n_1 = 17$

n_2	100 0	99.9 0.1	99.5 0.5	99 1	97.5 2.5	95 5	90 10	%→ ←%	10 90	5 95	2.5 97.5	1 99	0.5 99.5	0.1 99.9	0 100
17	152	210	223	230	240	249	259		336	346	355	365	372	385	443
18	152	214	228	235	246	255	266		346	357	366	377	384	398	460
19	152	219	234	241	252	262	273		356	367	377	388	395	410	477
20	152	223	239	246	258	268	280		366	378	388	400	407	423	494
%	0	0.2	1	2	5	10	20	⟷	20	10	5	2	1	0.2	0

$n_1 = 18$

n_2	100 0	99.9 0.1	99.5 0.5	99 1	97.5 2.5	95 5	90 10	%→ ←%	10 90	5 95	2.5 97.5	1 99	0.5 99.5	0.1 99.9	0 100
18	170	237	252	259	270	280	291		375	386	396	407	414	429	496
19	170	242	258	265	277	287	299		385	397	407	419	426	442	514
20	170	247	263	271	283	294	306		396	408	419	431	439	455	532
%	0	0.2	1	2	5	10	20	⟷	20	10	5	2	1	0.2	0

$n_1 = 19$

n_2	100 0	99.9 0.1	99.5 0.5	99 1	97.5 2.5	95 5	90 10	%→ ←%	10 90	5 95	2.5 97.5	1 99	0.5 99.5	0.1 99.9	0 100
19	189	267	283	291	303	313	325		416	428	438	450	458	474	552
20	189	272	289	297	309	320	333		427	440	451	463	471	488	571
%	0	0.2	1	2	5	10	20	⟷	20	10	5	2	1	0.2	0

$n_1 = 20$

n_2	100 0	99.9 0.1	99.5 0.5	99 1	97.5 2.5	95 5	90 10	%→ ←%	10 90	5 95	2.5 97.5	1 99	0.5 99.5	0.1 99.9	0 100
20	209	298	315	324	337	348	361		459	472	483	496	505	522	610
%	0	0.2	1	2	5	10	20	⟷	20	10	5	2	1	0.2	0

APPENDIX

Test Procedure for Large Samples

To develop a general test procedure, suppose that samples of sizes n_1 and n_2 are drawn from identical populations and are ranked jointly in the special manner described earlier. Then the sum of the $N = n_1 + n_2$ integers is $\frac{1}{2}N(N+1)$ and the sum of their squares is $\frac{1}{6}N(N+1)(2N+1)$. It follows that the mean and variance of the N integers are

$$\mu = \frac{1}{2}(N+1) \tag{1A}$$

$$\sigma^2 = \frac{1}{12}(N^2 - 1) \tag{2A}$$

If n_1 and n_2 are not too small, the means, R_1/n_1, of samples of size n_1 randomly drawn without replacement from the N integers will be approximately normally distributed with mean equal to μ and variance equal to

$$\sigma^2_{R_1/n_1} = \frac{\sigma^2}{n_1} \cdot \frac{N - n_1}{N - 1} \tag{3A}$$

and thus

$$z = \frac{R_1/n_1 - \mu}{\sigma_{R_1/n_1}} = \frac{R_1/n_1 - \frac{1}{2}(N \mp 1)}{\sqrt{(N^2-1)(N-n_1)/12n_1(N-1)}} \tag{4A}$$

will be approximately normally distributed with zero mean and unit variance.[3]

Transforming (4A) so that it is expressed in terms of R_1, the sum of the sample of n_1 observations, and correcting for continuity we have

$$z = \frac{2R_1 - n_1(n_1 + n_2 + 1) \pm 1}{\sqrt{n_1(n_1 + n_2 + 1)(n_2/3)}} \tag{1}$$

Correction for Tied Observations

If average ranks are assigned to tied observations, the mean $\mu = \frac{1}{2}(N+1)$ is not affected. However, the variance is reduced to less than $\frac{1}{2}(N^2-1)$. Since the variance is obtained by subtracting the square of the mean from the mean of the squares of the N consecutive integers, and since the sum of the average ranks is not changed, it follows that the change in variance depends only on the difference between the sum of squares of the ranks (not averaged), S_1, and the sum of squares of the averaged ranks, S_2. Thus the variance σ^2 is reduced by

$$\frac{S_1 - S_2}{N}$$

and becomes

[3] Mann and Whitney [14] show that as n_1, n_2 increase in size, the sampling distribution of the statistic U (which is linearly related to R_1) rapidly approaches the normal distribution.

$$\sigma^2 = \frac{1}{12}(N^2 - 1) - \frac{S_1 - S_2}{N} \tag{5A}$$

Substituting the right side of (5A) for the required population variance σ^2 in (3A) we get for the denominator of z

$$\sqrt{n_1(n_1 + n_2 + 1)(n_2/3) - [4n_1n_2/(n_1 + n_2)(n_1 + n_2 - 1)](S_1 - S_2)} \tag{3}$$

Procedure for Badly Balanced Samples

Two approaches to improving the approximation have been explored. The first, the Cornish-Fisher [4] form of the Edgeworth expansion, discussed by Kamat ([11], Chap. 6) is

$$z_{cf} = z + \frac{\gamma_2}{24}(z^3 - 3z) \tag{6A}$$

where γ_2 is a relative cumulant of the distribution of R_1. It is shown below that the distribution of R_1 has relative semi-invariants (relative cumulants),

$$\gamma_1 = 0$$

$$\gamma_2 = \frac{6}{5} \cdot \frac{n_2^2(n_2 + 1) - n_1^2(n_1 + 1)}{n_2n_1(n_2 + n_1 + 1)(n_2 - n_1)}.$$

The second approach is an empirical approximation, developed to roughly minimize the discrepancy between the desired (exact) values and the approximate values of the adjusted z's, namely

$$z_e = z + \left(\frac{1}{10n_1} - \frac{1}{10n_2}\right)(z^3 - 3z). \tag{2}$$

More refined approximations, in the original Edgeworth form giving corrections to the probability for a given value of z, are given by Fix and Hodges [7, pp. 312–3], who noted the great improvement due to the use of the first corrective term.

The adequacy of the two approximations considered here can be tested by applying them to the most extreme values for which the tables give exact significance levels, namely $n_1 = 2$ and 3 for $n_2 = 20$, and also for $n_2 = 10$. The results are shown in Table 2.

The quality of performance for these four extreme cases can be summarized as follows:

(1) The use of z itself for $|\text{exact } z| < 2.0$ leads to a maximum error of 0.09 and a mean absolute error of 0.03.

(2) The use of the empirical correction for $|\text{exact } z| < 2.5$, omitting the one instance $n_1 = 2$, $n_2 = 20$, $R_1 \leq 4$, leads to a maximum error of 0.05 and a mean absolute error of < 0.02.

Most situations outside the scope of the tables presented here will be adequately treated by the use of unadjusted z. In case of doubt the empirical adjustment should be sufficiently accurate except possibly for $n_1 = 2$, where the precise values are easily calculated.

TABLE 2. COMPARISONS OF THE APPROXIMATIONS WITH EXACT VALUES FOR SPECIFIED UNBALANCED SAMPLES

n_1 and n_2	R_1	Exact Probability	Corresponding z Values			
			Exact	Normal Approximation	Cornish-Fisher Approximation	Empirical Approximation
2 & 20	3	.00433	2.62	2.23	2.34	2.43
	4	.00865	2.38	2.11	2.19	2.25
	5	.0173	2.11	2.00	2.05	2.09
	6	.0260	1.98	1.89	1.92	1.94
	7	.0389	1.76	1.77	1.78	1.78
	8	.0519	1.63	1.66	1.66	1.64
	9	.0693	1.48	1.54	1.54	1.50
3 & 20	6	.000565	3.26	2.69	2.88	3.01
	7	.00113	3.05	2.60	2.77	2.87
	8	.00226	2.84	2.51	2.65	2.74
	9	.00395	2.66	2.42	2.54	2.61
	10	.00621	2.50	2.33	2.43	2.49
	11	.00903	2.36	2.24	2.32	2.36
	12	.0130	2.23	2.15	2.21	2.24
	13	.0175	2.11	2.05	2.09	2.12
	14	.0232	1.99	1.96	1.99	2.01
	15	.0299	1.88	1.87	1.89	1.90
	16	.0378	1.78	1.78	1.78	1.79
	17	.0469	1.68	1.69	1.69	1.68
	18	.0576	1.58	1.60	1.60	1.58
2 & 10	3	.0152	2.17	2.04	2.10	2.14
	4	.0303	1.88	1.83	1.85	1.85
	5	.0606	1.55	1.61	1.59	1.59
3 & 10	6	.003496	2.70	2.46	2.59	2.63
	7	.006993	2.46	2.29	2.38	2.41
	8	.01399	2.20	2.12	2.18	2.19
	9	.0245	1.97	1.95	1.98	1.99
	10	.0385	1.77	1.78	1.78	1.79
	11	.0559	1.59	1.61	1.60	1.59
	12	.0804	1.40	1.44	1.42	1.41

Moments and Cumulants

Mann and Whitney [14] give the second and fourth moments of the statistic U which is linearly related to R by

$$U = n_1 n_2 + \frac{n_1(n_1 + 1)}{2} - R.$$

They show that if

$$u = U - \frac{n_1 n_2}{2}$$

then

$$E(u^2) = \frac{n_1 n_2(n_1 + n_2 + 1)}{12}$$

and

$$E(u^4) = \frac{n_1 n_2(n_1 + n_2 + 1)}{240}(5n_2^2 n_1 + 5n_2 n_1^2 - 2n_2^2 - 2n_1^2 + 3n_2 n_1 - 2n_2 - 2n_1).$$

Thus the moment coefficients and relative cumulants are

$$\beta_1 = \gamma_1 = 0$$
$$\beta_2 = 3 + \gamma_2$$

where

$$\gamma_2 = \frac{k_4}{k_2^2} = \frac{E(u^4) - 3[E(u^2)]^2}{[E(u^2)]^2}$$
$$= \frac{6}{5} \cdot \frac{n_2^2(n_2 + 1) - n_1^2(n_1 + 1)}{n_2 n_1(n_2 + n_1 + 1)(n_2 - n_1)}.$$

(For an evaluation of the sixth moment see Fix and Hodges, [7, p. 312]).

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STATISTICAL EVALUATION OF CLOUD SEEDING OPERATIONS*†

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Modern commercial cloud seeding dates from the discovery in 1946 that silver iodide crystals are good nuclei for the formation of precipitation from cold clouds. Whether silver iodide seeding does actually increase the precipitation over what would occur naturally has been a controversial question.

In 1953 Congress established the Advisory Committee on Weather Control, one of its responsibilities being to determine whether the United States should experiment with activities designed to control weather conditions. In its Final Report, 1957, the Committee concluded on the basis of historical records that in mountainous areas commercial seeding produced an increase of 10 to 15 per cent in precipitation. The methodology of the Committee's analysis is reviewed in this paper, and it is concluded that possible selectivity on the part of the seeding operator could produce fictitious results. The Committee's Report denies the existence of such selectivity but there is reason to doubt the denial. In the reviewer's opinion the Committee's claim for the positive effects of commercial seeding must be regarded as unproved.

Since man's earliest civilizations he has on occasion wished to increase the rainfall to increase the yield of his crops. In fact, entire civilizations have crumbled in consequence of a secular decline in rainfall. In our present civilization, weather modification, if it is possible, would be of great economic importance. For example, hurricanes, droughts, floods, and hail storms cause economic loss. Those who live or own property in low-lying territory liable to flood would be happy if floods could be prevented, those raising crops in areas of notoriously inadequate rainfall would like the precipitation increased. In some instances we can affect the weather by sheer brute force. Fog has been dispersed locally by the burning of fuel oil to permit the landing of military aircraft. Presumably, exploding a large atomic bomb in a small thunderstorm would bring that thunderstorm to a close. However, what is relevant is whether changes in the weather can be made economically, in the sense that the benefits outweigh the costs.

Modern interest in weather modification, specifically increase in precipitation, dates from 1946 when V. J. Schaefer dropped dry ice into supercooled clouds over Massachusetts and observed that holes were formed in the clouds. In the same year, B. Vonnegut showed that in laboratory cold boxes silver iodide crystals were good nuclei for the formation of ice crystals. The relevance of this is that, according to the Bergeron-Findeisen theory, the formation of ice crystals is an essential step in the formation of precipitation from cold clouds. A large amount of government-sponsored research followed. By 1948 commer-

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cial operators claiming to increase precipitation, usually with ground based silver iodide generators, were in business. Their claims were disputed by some meteorologists and in 1953 Congress set up the Advisory Committee on Weather Control. The Committee consisted of representatives of the Departments of Agriculture, Commerce, Defense, Health (Education, and Welfare), and the Interior, and the National Science Foundation, along with five "persons in private life of outstanding ability in the fields of science, agriculture, and business." Section 3 of the act reads:

"The Committee shall make a complete study and evaluation of public and private experiments in weather control for the purpose of determining the extent to which the United States should experiment with, engage in, or regulate activities designed to control weather conditions."

The two volumes under review here form the Committee's Final Report.

Volume I contains a Summary of Findings, of which the most important is that the Committee concluded:

"(1) The statistical procedures employed indicated that the seeding of winter-type storm clouds in mountainous areas in western United States produced an average increase in precipitation of 10 to 15 per cent from unseeded storms with heavy odds that this increase was not the result of natural variations in the amount of rainfall.

(2) In nonmountainous areas, the same statistical procedures did not detect any increase in precipitation that could be attributed to cloud seeding. This does not mean that effects may not have been produced. The greater variability of rainfall patterns in nonmountainous areas made the techniques less sensitive . . .

(4) Available hail frequency data were completely inadequate for evaluation purposes . . . "

The Recommendations of the Committee do not include any recommendation that one or more properly designed and excuted experiments should be run to test whether commercial seeding techniques have any effect in increasing precipitation. Since in its Findings the Committee had already concluded that commercial seeding techniques are effective in mountainous areas, they might have felt that such a recommendation for mountainous areas was inappropriate, but their findings had specifically given no conclusion for nonmountainous areas and it might have been justifiable to recommend further work in such areas.

The remainder of Volume I summarizes the historical background, the activities of the Committee, and the papers on the statistical and physical evaluation which form the bulk of Volume II and which are the basis for the Findings quoted above.

In this paper we are not directly concerned with the fifteen papers in Volume II on the physical evaluation program, nor with the paper on the legal situation. The section on the statistical evaluation program includes six technical reports. The most important from a statistical and practical viewpoint are Nos. 1 and 2, both by H. C. S. Thom, on "A Statistical Method of Evaluating Augmentation of Precipitation by Cloud Seeding," which sets forth an analysis closely related to a covariance analysis, and "An Evaluation of a Series of

Orographic Cloud Seeding Operations," which applies the techniques of Report No. 1 to several sets of historical data. These reports will be discussed later in this review. Report No. 3, "The Frequency of Hail Occurrence," by H. C. S. Thom, concludes that the distribution of the occurrence of hail follows a negative binomial distribution and discusses the estimation of the parameters. Report No. 4, "A Method for the Evaluation of Hail Suppression," by H. C. S. Thom, develops a sequential test for the situation in which the underlying distribution is the negative binomial. No. 5 by Gerald D. Berndt, on "An Evaluation of Commercial Cloud Seeding Operations Conducted During the Summer Months in South Dakota" introduces the concept of a cost/benefit ratio and a Bayes approach. It appears to use the average rain for those days in which rain did fall, rather than for all relevant days. Thus, if seeding completely suppressed rain on certain days, this effect would not show up in the average considered. No. 6, by Max A. Woodbury, is concerned with prediction of what would happen in the absence of seeding, using relative humidity and similar variables. For monthly precipitation in South Dakota relative humidity gave a correlation coefficient of 0.614 and the addition of two more sophisticated variables raised this to 0.623.

Volume II also includes a number of miscellaneous items. Appendix A reproduces the Act of Congress establishing the Committee. Appendix B is largely a diary of the movements of the staff of the Committee. Appendix C is two pages on "The Physical Basis of Cloud Modification" by F. A. Berry, Chief Scientific Advisor to the Committee. Appendix D is a bibliography of 2259 items. The value of this bibliography is lessened by the inclusion of much trivia in such media as the *Washington Post Parade Magazine* or *Colliers*. As a check on the accuracy of this bibliography the reviewer took a sample of one, admittedly biased, and looked up his own name in the author index. His name was therein as paper number 173, but reference to that number showed another name. The reviewer was also unable to find any reference to the important mimeographed Report No. 24, "Artificial Nucleation of Cumulus Clouds," by Roscoe R. Braham, Jr., Louis J. Battan, and Horace R. Byers. At the time the Final Report was written this experiment was almost unique in being a carefully designed and executed randomized experiment on cloud modification. Amongst the tiny class of completed randomized experiments it was unique in showing a statistically significant positive result for the rather special circumstances and seeding technique involved. The failure of the Final Report to give an explicit account of this work is extraordinary.

The general lines of the appropriate statistical methodology for testing the null hypothesis that seeding is without effect appear pretty clear. If a particular site, denoted by the "target," is seeded at every opportunity, then the estimated rain which falls must be compared with an estimate of the rain which would have fallen in the absence of seeding. One way of getting such an estimate is to take the historical average. This proves unsatisfactory for a number of reasons:

(a) The rain gauge system used in the past is frequently very inadequate for this purpose. Any improved rain gauge system installed to check on the effects of seeding will not be comparable with the historical gauge system.

(b) The variability of rain at one site over time is so great that enormous

numbers of observations are needed to give reasonable probability of detecting a reasonable change.

(c) The assumption that the mean rainfall at a given site is unchanged is open to question. It seems clear that long and short term changes do take place.

The design can be greatly improved by using another site as a "control," and making an analysis of covariance using the rainfall on the control, say x, as the concomitant variate for the adjustment of the rainfall on the target, y. The control must not be so close to the target that it is accidentally seeded when the target is seeded. On the other hand, to be of any use in reducing the variance of y given x, $V[y|x]$, the control must be close enough to have a good correlation with target.

In a satisfactory experiment, a workable and objective definition of the "experimental unit" must be determined. This might be an arbitrary period, for example, a week, or a meteorological unit, for example, the passage of a high pressure ridge over a meridian, or a "storm."

It is frequently difficult to formulate a completely satisfactory definition of unit. An arbitrary definition such as a day or week satisfies the requirement of objectivity, but it seems unsatisfactory to shut down the seeding generators in the middle of a storm and attribute all rain that fell after that moment to an unseeded period. Dummy periods could be inserted routinely between experimental periods but this will waste time which is precious. With natural meteorological units, depending upon the criteria for the unit there may be danger of subjectivity in deciding when a particular unit has ended. For the experiment to be valid either the criteria must be absolutely objective or, if the criteria contain some subjectivity, the decision that a particular unit has ended must be made by a person in complete ignorance of whether that unit was seeded or unseeded. On the one hand, completely objective criteria drawn up before the start of the experiment may subsequently encounter unanticipated circumstances which make the criteria look rather silly: on the other hand, ignorance of a member of the research team is difficult to maintain. The determination of a satisfactory unit is usually by no means easy.

In any actual experiment a choice must be made of the seeding technique. In view of the claims made for ground based silver iodide generators, an experiment using this equipment in a similar mountainous area would be desirable. Another experiment could be to seed at a very high rate directly into clouds from one or more airplanes. Conceivably this latter technique, though more expensive to operate, might be more effective.

Assuming that the seeding technique has been selected and the experimental unit defined, as each unit occurs a randomized choice is made whether to seed or not to seed. The observations will then be y_{ij} for the target and x_{ij} for the control, $i=1$ for the unseeded units and $i=2$ for the seeded units; the index j refers to the sequence of the units in each class. A natural statistical analysis is to compare $\bar{y}_{1.}$ with $\bar{y}_{2.}$, adjusting for covariance on the x_{ij}. This, of course, is equivalent to fitting straight lines to the two classes of data, testing for parallelism, and then testing for coincidence of the lines.

This is essentially the procedure followed by Jeeves, LeCam, Neyman, and Scott, reported in 1952 and 1953, and published in 1955 [1]. Thom, in Report

No. 1 in Volume II, published in 1957, followed the same procedure in broad outline but with some modifications.

The analysis requires homoscedasticity, i.e., that $V[y|x]$ should be a constant, and also the analysis is much more convenient if the regression of y on x is linear. The California group, Jeeves *et al.*, used a square root transformation of both variables. Others have found a logarithmic transformation satisfactory. Thom uses a more complicated transformation. He claims that the rainfall at a given site has a gamma distribution. He then estimates the parameters of these distributions and transforms the actual observations into a unit normal distribution. Since all that is required for the covariance analysis is normality and homoscedasticity in the residuals about the regression lines, and these appear to be closely achieved by the square root or logarithmic transformations, this step appears to be unnecessarily elaborate, but not harmful.

Thom also remarks that, "Since there is a small correlation between successive storms but practically none between alternate storms, our recommended practice is to fit odd and even storms in a period separately and combine the two estimates by averaging." What is relevant, of course, is correlation between the *residuals* and it is not clear whether this remark is referring to residuals or the storms themselves.

Thom also differs from standard practice in fitting a line to only the unseeded set of data and his test statistic is $\bar{d}_k/\sqrt{\hat{V}[\bar{d}_k]}$, where $\bar{d}_k$ is the mean of the deviations of k seeded points from the values predicted by the estimated historical regression line. No explanation is offered for this asymmetric treatment of the two classes of points. If, in point of fact, the vast majority of points are in the unseeded class, as in the data to which Thom does apply this analysis, then there will be little difference between this procedure and the standard one of fitting parallel lines to the two classes. On the other hand, if the sizes of the two classes are roughly equal there will be some inefficiency in Thom's analysis. Also, if there is nonparallelism of the regression lines, this would indicate that seeding has an effect which varies according to the intensity of the storm. If the effect is negative for small x and positive for large x, or vice versa, the situation could occur in which the effect averages to zero and a consideration of $\bar{d}_k$ would not show anything.

A further defect of Thom's procedure is that it uses the variance of the individual historical points about the historical regression line as an estimate of $V[y|x]$ which is then inserted in the formula for $\hat{V}[\bar{d}_k]$. It is possible that the seeded data has a larger (or smaller) variance than the historical data which is unseeded. The standard analysis, of course, includes the estimation of $V[y|x]$ separately for seeded and unseeded data and permits a check on the null hypothesis of their equality.

Most data appear to conform well with the assumptions of the standard covariance analysis and there does not seem to be much need to go to the analogous nonparametric procedures which have been described.

Parenthetically, a more sophisticated design has been proposed in which two areas are observed, and randomly one or the other is seeded, thus giving an analog to a cross-over design. At the time of the Committee's Final Report such a design had not been used anywhere, and we will not discuss it further.

The foregoing discussion has presupposed a properly randomized experiment, in which there would be two treatments applied to the target, seeding and not-seeding. Unfortunately, at the time the Committee was established there had been very few, if any, experiments, in the sense in which a statistician would accept the word. The Committee did feel free to sponsor experimental work on "what takes place in clouds when silver iodide crystals are released at the ground or from aircraft," in fact they were responsible for what they called Project Overseed, Project Sailplane, Project Skyfire, Arizona, 1956, Project Skyfire, Montana, 1956, Project Seabreeze, and Project Skyfire, Montana, 1957, but they took the position that they could not run or sponsor any true experiments on rain making:

"From the outset the Committee was well aware that within its statutory authorization it could undertake a program of only limited scope:

"(1) It could not, for lack of sufficient time, carry on long-range research, desirable if not essential to evaluation, such as scientifically designed, randomized cloud seeding experiments. It had, therefore, no alternative but to employ available 'historical' data as the material for evaluation studies."

Public Law 256 of the 83d Congress, approved August 13, 1953, required the Committee to submit a final report not later than June 30, 1956. This was amended to June 30, 1958 on July 9, 1956. It is clear that Congress, in handing the Committee the task of making "a complete study and evaluation of public and private experiments," had given them a futile assignment when in fact at that time there were virtually no experiments available to evaluate.

The Committee chose to put a substantial effort into the evaluation of a number of commercial projects and the associated historical data (Report No. 2 of Volume II). These were divided into three groups, one about mountainous areas on the west coast, the second about semi-mountainous areas, and the third about non-mountainous areas. For the first group they found that the seeded data showed an increase of 14 per cent over the historical expectation, a result significant at the 0.01 level of significance. The Committee used this as the basis of their main conclusion, that seeding in this type of area produced an average increase in precipitation of 10 to 15 per cent.

If this analysis had been based on data arising from properly randomized experiments this conclusion would be acceptable, but, since it is based on non-randomized data, it needs careful scrutiny.

Jeeves *et al.* [1] had remarked that in the Carrizo Plain area storms could be divided into several types, and in general the regression lines for the different types were not the same. Thus if the proportions of these types in the later data were different from the proportions in the earlier data, then there would be a difference between the two sets of data. If the later data corresponded to seeded data, the change might be attributable to the effect of seeding whereas in reality it might be due to this change in proportions of types.

Furthermore, if the seeder, consciously or subconsciously, chooses to seed only storms of those types for which the target shows up favorably compared to the control, then he could produce a fictitious positive effect of seeding.

Thom discusses this possibility and states, "As to the operator's seeding of a nonrandom selection of storm types, it may be stated categorically that he did

not." For the operator to work this device effectively, he would have to guess which control area would be used. Clearly each case would have to be examined carefully to see whether in fact there were plausible alternative control areas. The operator would also have to select storms for seeding, which implies that some storms would go unseeded. Thom remarks, "Insofar as we could determine, they (the operators) seeded every storm except those occurring in periods when they were specifically instructed not to seed by their clients. They were quite skilful in seeding all allowable storms for there were very few left unseeded in the storm series." This is clear enough, but Neyman, Scott, and Vasilevskis [2] reviewed commercial operations by nine operators in California in 1951–2 and found that for those days under contract in which more than 0.10 inches of rain fell, 41 per cent were not seeded. These figures and Thom's assertion appear inconsistent.

The potential importance of this phenomenon of the seeder choosing only to seed a fraction of the seeding opportunities is illustrated by two scatter diagrams of Neyman *et al.* [2]. In an experiment in 1957–59 in Santa Barbara, a commercial operator was to report occasions suitable for seeding to a "randomizer" who would then tell him whether or not to actually seed the target. It turned out that the seeder reported as seeding opportunities 51 per cent of those 12-hour units of observation in which more than 0.10 inches of rain fell on the target. Those occasions he diagnosed as suitable for seeding gave a relatively high precipitation on the target compared with the control, whereas those occasions which he failed to diagnose as suitable for seeding gave a relatively low precipitation on the target compared with the control. Selectivity of this nature would give a fictitious positive effect for seeding. In the present experiment we are able to observe it, since half the occasions declared suitable for seeding were not seeded, through the chance effect of the randomization procedure. Since half the storms suitable for seeding were in fact not seeded, we can observe the effect of selection without the picture being confused by the presence or absence of seeding, and we observe that the effect of selectivity is to favor the target.

To belabor this point, since it is crucial to the entire question, we have two groups of storms, those declared suitable for seeding and those declared unsuitable. In the present experiment half of the former group were left unseeded, and so can be compared directly with the latter group: they are different, favoring the target. In ordinary seeding operations, on the other hand, all the former group would be seeded. They would be compared with the latter group, and found to be different, the target having relatively large amounts of rain. This increase would be attributed to the seeding which in ordinary seeding operations would have been performed. From the randomized experiment, we see that this increase is completely attributable to the selectivity.

It is clear that the Committee's analysis of the historical data leaves unsettled the basic question of whether seeding of the type used in the commercial operations under analysis has any effect on precipitation. The Committee's main conclusion is thus unjustified. It is surprising that they were prepared to make this conclusion, for its invalidity was clearly set out in 1955 in [1], and furthermore after the Committee stated this conclusion in their First Interim

Report on February 8, 1956, they were strongly criticized by several speakers at the Conference on the Scientific Basis of Weather Modification Studies at the University of Tucson in April 1956.

The Committee was hardly a shoe-string operation. For fiscal 1955, 1956, and 1957, $120,000, $275,000, and $300,000 were appropriated. For fiscal 1958 the Senate Appropriations Committee recommended $250,000, but the House denied further funds. On July 1, 1957 the Committee was without funds, but on August 28, $100,000 was available for use before the end of the calendar year for preparing and publishing the Final Report.

It is frustrating to contemplate the possibility that the above money, $795,000, if devoted to one or two deliberate, well-planned and executed experiments, might have settled the question. As it is, the emotions generated by this topic make the carrying out of a satisfactory experiment a Herculean task. Apart from the considerable scientific and logistic problems, scrupulous attention to the possibility of bias, conscious or subconscious, and fraud is necessary. The question of whether it is necessary to station guards at the gauges to prevent "salting" by interested parties needs serious examination. The Final Report of the Advisory Committee on Weather Control has settled none of the important issues, and its failure points up the need for a truly objective determination of these questions.

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THE USE OF STATISTICS IN THE FORMULATION AND EVALUATION OF SOCIAL PROGRAMMES

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Social statistics are defined as the statistics required for social programmes, and a social programme as the scheme of public and private activities which have a direct bearing on a particular aspect of living conditions. It is assumed that there is a separate programme for each component of the levels of living, e.g. health, education, housing.

The uses of statistics for the following aspects of social programming are examined and illustrated with examples from the field of housing: estimating of economic resources for the programme; measurement of living conditions by statistical indicators; estimating of social needs, e.g. "dwellings," to maintain or improve the levels of living of growing populations; testing of the consistency of social needs and economic, human and material resources; and adoption of goals for social programmes consistent with expected resources.

Several forms of evaluation are briefly discussed, and the indicators of levels of living are considered to be the only comprehensive yardsticks of the success or failure of social programmes.

The purpose of the paper is to provoke dissemination of information on this subject among under-developed countries and to stimulate research on the uses of statistics for social programming.

1. INTRODUCTION

THE role of statistics in guiding decisions regarding micro aspects of social programmes seems to be well established. But, the possibility of using statistics in formulating social programmes becomes uncertain as the scope of such aspects increases, and the question of whether and how statistics can be used in formulating social programmes at the national level has no obvious answer. As a result, while it may be assumed that minor decisions in the social field can be, and frequently are, based on scientific information, the major decisions regarding social policy and programmes—especially in under-developed countries—are mainly a matter of personal judgement based primarily on political expediency and traditional grounds.

The purpose of this paper is to explore the possibility of using social statistics in formulating national social programmes, to provoke wider dissemination of information on this subject in the under-developed countries and to stimulate further research on the methodology of formulating national social programmes.

The discussion here has been restricted in some cases to the field of housing, but it is hoped that similar applications may be found in other social fields.

2. SOCIAL STATISTICS AND SOCIAL PROGRAMMES

The first problem that confronts anyone discussing social statistics is to decide what meaning to attach to the concept. In current usage the scope of

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social statistics ranges from a narrow coverage which includes only one aspect of social conditions, such as crime or the volume of work of welfare institutions, to the broadest possible range of subjects related to the description of the social structure, social institutions, human relations, environmental conditions, and government and private activities affecting living conditions. It seems that in spite of the attempts that have been made, social statistics cannot be identified as a unitary field by reference to common units of observation, or methods of collecting and analysing data. However, "social statistics" undefined have been the subject of continued attention in recent years, and this may be sufficient justification for preserving the expression (social statistics), defining in each case the meaning attached to it.

For the purpose of this discussion social statistics are defined as the statistics required for the formulation and evaluation of broad national social programmes. This definition focuses attention on the use or purpose of the statistics rather than on the nature of the units of observation or on the methods of collecting and analysing the data. Though non-rigorous, this definition is useful since it stresses the principle that statistics, and government statistics in particular, must always have a well defined purpose.

Little would be gained, however, by defining social statistics as the statistics required for social programmes unless social programmes can be defined independently of social statistics. A social programme may be considered to be formed by the scheme of public and private activities which would contribute to maintain or improve some particular aspect of the living conditions. The aim of such a programme would be to raise the levels of living at the maximum practicable rate commensurate with long-term economic development and human resources, and in accordance with nationally and internationally accepted principles regarding human rights and responsibilities.

It may be recalled that, in 1953, a United Nations Committee of Experts on International Definition and Measurement of Standards and Levels of Living recommended the use of twelve separate components of the levels of living,[1] each component representing a well delimited aspect of the levels of living (i.e. of the actual living conditions). In view of this recommendation and of the fact that aims of social policy are usually stated with reference to particular fields, such as education, health, housing and nutrition, it will be assumed, for the purpose of this discussion, that a separate social programme will be formulated for each component of the levels of living. Such an assumption should be acceptable since the techniques and factors involved in improving living conditions in each field are different and separate authorities or government agencies are responsible for the conduct of the various programmes.

The activities involved in the various programmes are of a different nature and, therefore, the methods of formulating the programmes are also quite different. Housing programmes, for example, are concerned with maintaining an adequate supply and distribution of dwellings; education programmes with maintaining an adequate supply and distribution of teaching services and facilities; nutrition programmes with promoting adequate food consumption.

[1] United Nations, *Report on International Definition and Measurement of Standards and Levels of Living*, Document No. E/CN.3/179-E/CN.5299, New York: United Nations, Sales No. 1954.IV.5.

Health programmes are conceptually more complex since the levels of health[2] depend upon a great many factors including housing, education, nutrition and so forth. Some social programmes are concerned with the supply and distribution of certain essential goods or services; others with the prevalence of certain conditions such as social security or human freedoms. Statistics may be useful in connexion with every type of programme but the uses will differ according to the characteristics of each programme.

Although it is expedient to consider each social programme separately, it must be recognized that they tend to be highly interrelated in the sense that the execution of one programme has direct implications for the living conditions in other social fields, e.g. malaria eradication, food supply and labour force; improvement of levels of education and housing conditions and levels of health; sanitation and nutrition, etc. Furthermore, there is an evident inter-action between social programmes and economic development. However, there is no mechanism for integrating social programmes in terms of activities and units of "well-being," such as is found in the economic area in the form of a system of national accounts, input-output tables and mathematical models, which aggregate and relate economic actions in terms of monetary units, and it is, therefore, necessary to deal with each social programme separately.

3. USES OF STATISTICS IN FORMULATING A SOCIAL PROGRAMME

The task of formulating a national social programme does not consist in solving a system of equations. Mathematical models describing the effect of social measures on the levels of living, the cost of such measures and the physical inputs involved have not been established. Such models would be useful, even if based solely on non-tested hypotheses and informal estimates, since they would set forth in an explicit form the "laws" or the assumptions involved in formulating the programmes. Unfortunately, complex and complete hypotheses, models, or laws describing the interaction of demographic, social and economic factors seldom exist, and the formulation of national social programmes can only be based on simple hypotheses, hardly more dignified than crude hunches. However, carefully selected statistics can materially enlighten the bases for the decisions by providing the factual information required to ensure compatibility between social needs and resources, and the statistics recording the experience of previous years will provide perhaps the most valuable guide to planners.

From a practical point of view, estimates of the economic resources, i.e. expenditure and investment by the public and private sectors, provide a suitable point of departure for the formulation of short term programmes in a particular social field. As a next step, the volume of public and private activities needed in order to maintain or improve the levels of living might be estimated and the total value calculated by using unit costs. The cost of the national programme could then be compared with the resources. If the resources appear to be insufficient to maintain or moderately raise the levels of living of a particular

[2] Health is defined in the constitution of the World Health Organization as follows: "Health is a state of complete physical, mental and social well-being and not merely the absence of disease or infirmity." See: United Nations, *Final Acts of the International Health Conference*, held in New York from 19 June to 22 July 1946, Document No. E/155, New York: United Nations, October 1946.

component, a revision of the resource allocation would need to be considered, together with the possibility of improving the efficiency and productivity of the agencies and industries involved by introducing new methods or procedures. After all possibilities are explored—and statistics would be essential for such an examination—a social programme could be outlined which would indicate the volume of activities that would be likely to be performed and the probable consequences of the programme upon the levels of living.

It is true that there are other limiting factors of social programmes, besides the economic resources, which must also be duly considered, but the system of economic statistics provides a partial linkage of great operational significance between economic development and social programmes. In effect, for practical purposes and considering only short term programmes, capitalist and socialist countries alike consider social development as a by-product of economic development, and resources for social programmes are estimated or allocated on the assumption that adequate provision is made for ensuring—as a matter of first priority—an adequate rate of economic development[3] and a proper balance between consumption and production, in order to avoid inflationary pressures.

On the assumption that the process broadly outlined above would be followed in formulating a social programme, statistics would be required for the following major aspects of the programming operation:

a. To estimate the economic resources expected to be available for the social programme,
b. To measure living conditions by means of indicators of the levels of living,
c. To estimate the social needs, i.e. the supply and distribution of "social units," required for maintaining or improving the levels of living,
d. To test the compatibility of the estimated social needs with the expected economic and human resources,
e. To select the most convenient combination of activities, i.e. a social programme.

Uses of statistics to estimate economic resources expected to be available for the social programme 3.1

Considerable attention has been given to the application of statistics in connexion with problems of economic planning and development. Principal sources of information regarding expenditure and investment in previous years are the national accounts[4] and the government accounts[5] when these systems provide separate data for the several social programmes. It is not, however, an easy task to separate out the data for various categories and eventually obtain separate totals for the country as a whole. For example, the apportionment of certain items of public expenditure among related social fields involves problems of interpretation, such as the allocation of school feeding programmes to health or to education; the allocation of urban water supply projects to health, hous-

[3] In capitalist countries by means of an adequate rate of investment in the means of production. In socialist economies by securing a rate of increase of the output of means of production greater than the rate of increase of the output of consumers' goods.

[4] United Nations, *A System of National Accounts and Supporting Tables*, Document No. ST/STAT/Series F./No.2/Rev.1, New York: United Nations, Sales No. 1959.XVII.11.

[5] United Nations, *A Manual for Economic and Functional Classification of Government Transactions*, Document No. ST/TAA/M/12—ST/ECA/49, New York: United Nations, Sales No. 58.XVI.2.

ing, or other programmes; the allocation of the cost of marsh clearance to health or to agriculture, and so forth. Some of the statistical data commonly used in evaluating the economic resources devoted to housing programmes, are the following:[6]

a. *Expenditure for dwelling construction.* This item, from the National Accounts includes all expenditure on new dwelling construction and major alterations to residential buildings including the value of the change in work, in progress, the cost of painting and all permanent fixtures, but excluding the value of the land before improvement,
b. *Rent and water charges.* This item of private consumption expenditure includes all gross rent (actual and imputed gross rent on owner-occupied houses as well as actual and imputed grounds rents payable) including water charges and local rates.

These items provide information about the order of magnitude of the economic resources devoted to housing in the past. When related to the national income and other national aggregates they provide ratios which are useful in estimating future resources, on the assumption that past patterns of expenditure and allocation would prevail or change in a foreseeable direction. The following ratios may be used for this purpose:

a. Expenditure on dwelling construction as a percentage of gross domestic product,
b. Expenditure on dwelling construction as a percentage of gross fixed capital formation,
c. Rent as a percentage of private consumption expenditure.

Estimates of the gross domestic product, private consumption expenditure and fixed capital formation, over the period to be covered by the programme, will usually be obtained by the agencies forecasting economic development. On the basis of these estimates it would be possible, by applying assumed ratios based on past experience and other factors, to derive estimates of the order of magnitude of the national resources that are likely to be devoted to housing in the next few years. Such estimates, would, of course, need to be revised every year, in the light of new information. This procedure is a very crude one but as crude as it is it may be considered too elaborate for many underdeveloped countries.

As an additional guide in estimating resources it may be useful to compare the situation or patterns of expenditure in various countries. Table 1 shows the values of the three ratios described above for 32 countries, in 1957 or 1956. The variation among countries is considerable and apparently there is no relation between the ratios and the state or rate of development of the countries. Also, the ratios in the countries with long historical data available show long term and cyclical variations as well as irregular changes. International comparisons, therefore, show only the range of variation of these ratios but would

[6] The evaluation of the economic significance of housing programmes is discussed in United Nations, Administrative Committee on Co-ordination, *Statistical Indicators of Housing Levels of Living*, a methodological study of a preliminary nature, Document No. ACC/WPSSP/I/4/Add.6, New York: United Nations, 6 August 1959.

be rather unhelpful for the purposes of selecting suitable rates of expenditure investment in housing. But long national series would be extremely useful guides in determining the resources that are likely to become available, over a period of a few years, for housing as well as for other social programmes.

An important factor in connexion with estimating the economic resources available for social programmes is the distribution of resources between the public and private sectors. In some countries it has been found to be more efficient to finance important aspects of social programmes through government

TABLE 1. EXPENDITURE FOR DWELLING CONSTRUCTION AS A PERCENTAGE OF GROSS DOMESTIC PRODUCTS AND GROSS FIXED CAPITAL FORMATION, AND RENT AS A PERCENTAGE OF PRIVATE CONSUMPTION EXPENDITURE.* SELECTED COUNTRIES. 1956 OR 1957

Country	Year	Expenditure for dwelling construction		Rent as a % of PCE
		% of GDP	% of GFCF	
Australia	1957	—	—	8.4
Austria	1957	—	—	4.8
Belgium	1957	4.1	25.3	13.0
Canada	1957	4.5	16.7	15.4
Ceylon	1957	—	—	2.7
Denmark	1957	2.9	16.8	6.4
Dominican Republic	1957	—	—	11.1
Ecuador	1956	2.6	19.0	10.5
Finland	1957	—	—	6.4
France	1957	4.9	26.1	4.6
Ghana	1957	—	—	2.0
Greece	1957	4.8	30.5	—
Ireland	1957	2.2	16.9	4.0
Israel	1956	7.1	29.7	—
Italy	1957	5.9	27.4	3.1
Japan	1957	2.1	7.9	9.0
Korea, Rep. of	1957	0.5	4.1	7.9
Luxembourg	1957	4.0	17.2	—
Mauritius	1957	3.0	23.1	—
Morocco	1956	3.0	24.5	—
Netherlands	1957	5.5	22.0	6.9
New Zealand	1957	4.8	21.7	—
Nigeria, Fed. of	1956	2.1	20.7	—
Norway	1957	4.4	15.7	7.6
Panama	1956	—	—	16.4
Peru	1956	—	—	20.3
Philippines	1957	1.6	18.4	—
Portugal	1957	4.1	25.7	—
Puerto Rico	1957	4.2	19.1	10.5
Sweden	1957	4.8	24.2	9.2
United Kingdom	1957	2.9	18.2	8.6
United States	1957	4.2	23.7	12.6

* At current market prices.

Source: United Nations, Administrative Committee on Co-ordination, *Statistical Indicators of Housing Levels of Living*, Document No. ACC/WPSSP/I/4/Add.6, New York: United Nations, 6 August 1959.

agencies, while in others the direct purchasing of essential goods and services by the individual households is considered possible and satisfactory. The general situation may be considered to be a mixed arrangement by which programmes are financed through both the public and private sectors. It is important, therefore, to have information about the extent to which the various programmes are financed by each of the two sectors. This question was discussed in a paper[7] presented to the fifth session of the ECAFE Working Party on Economic Development and Planning, held in Bangkok, in September 1959. Table 2, which was taken from this paper, illustrates how data on resources

TABLE 2. PROVISION AND FINANCING OF EDUCATIONAL SERVICES (IN MILLIONS OF NATIONAL CURRENCY)

Provided by: \ Financed by:		Public sector		Private sector		Total
		Central government	Provincial and local governments	Households	Enterprises	Expenditure for providing educational services
Public sector	Central government	*a*	—	*b*	—	
	Provincial and local governments	*a*	*a*	*b*	—	
Private sector	Non-profit institutions	*b*	*b*	*a*	—	
	Enterprises	—	—	*b*	*b*	
Total	Expenditure for financing educational services					

a = likely to be significant
b = moderate significance
— = negligible
} Notes applicable to ECAFE countries.

Source: United Nations, Economic Commission for Asia and the Far East, Working Party on Economic Development and Planning, *Planning the Pattern of Public Social Expenditures*, Document No. E/CN.11/DPWP.5/L.7, New York: United Nations, 4 August 1959.

would need to be broken down by sectors for educational programmes. A similar table could be prepared for other programmes.

A further breakdown of data on resources according to investment or current expenditure would also be of interest. The *Manual for Economic and Functional Classification of Government Transactions*[8] contains a full discussion of this aspect.

Use of statistics to measure living conditions by means of indicators of the levels of living 3.2

The purpose of social programmes is to raise the levels of living, and, in order to translate this purpose into positive action, it is necessary to express levels

[7] United Nations, Economic Commission for Asia and the Far East, Working Party on Economic Development and Planning, *Planning the Pattern of Public Social Expenditures*, Document No. E/CN.11/—DPWP.5/L.7, New York: United Nations, 4 August 1959.

[8] *Op. cit.*

of living in statistical terms. The United Nations Committee of Experts, mentioned above, proposed in 1953, that the levels of living should be measured by reference to twelve components by means of a series of statistical indicators for each component. Recently, in 1959, an inter-agency Working Party on Statistics for Social Programmes[9] reviewed the components and indicators in the light of information that had become available since 1953, reduced the twelve components to nine, and proposed the following priority indicators:

Component Health:

1. Expectation of life at birth,
2. Infant mortality rate,
3. Crude annual death rate.

Component Food Consumption and Nutrition:

1. National average food supplied, in terms of calories at the retail level, compared with estimated calories requirement,
2. National average food supplied, in terms of total proteins at the retail level,
3. National average food supplied, in terms of animal protein at the retail level,
4. Per cent of total calories derived from cereals, roots, tubers and sugars.

Component Education:

1. Adult literacy rate,
2. Total school enrolment ratio,
3. Higher education enrolment ratio.

Component Employment and Labour Conditions:

1. Proportion of persons employed in the total labour force,
2. Ratio of male labour force in agriculture to total labour force,
3. Relative real wages in selected occupations.

Component Housing:

1. Per cent of the population living in "dwellings",
2. Per cent of occupied dwellings with three or more persons per room (overcrowding),
3. Per cent of occupied urban dwellings with piped water.

Component Social Security:

[No indicators proposed as yet.]

Component Clothing:

[No indicators proposed as yet.]

Component Recreation:

[No indicators proposed as yet.]

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[9] United Nations, Administrative Committee on Co-ordination, *First Technical Report of the Working Party on Statistics for Social Programmes* (Geneva, 16–22 September 1959), Document No. ACC/WPSSP/I/9/Rev.1, New York: United Nations, 22 October 1959.

Component Human Freedoms:

[No indicators proposed as yet.]

The indicators are of various classes; some reflect achievement or ultimate aims of social programmes in very broad terms, as in the case of the expectation of life at birth; others are more directly related to the activities in the respective social programme, as in the case of the total school enrolment ratio. The indicators of housing levels of living belong to the first class. They are operational indicators which define aims of social programmes and they can also be utilized for estimating the order of magnitude of the programme needed in order to improve the living conditions in respect of housing. Some of the other indicators are not so useful for formulating programmes because, though reflecting certain aspects of living conditions, their relationship to the possible actions in a particular social field are not well established. This seems to be the case with respect to the indicators of health which are not directly related to actions in the medical fields. For example, at present it appears impossible to determine what type of medical or health activities would be necessary in order to increase the expectation of life at birth by one year or reduce the infant mortality rate by five units, or the crude death rate by one unit. On the other hand, the relationship between indicators of housing or education conditions and actions in these fields are understood and can be utilized in formulating relevant programmes. There is much to be done in developing methods of measuring levels of living and particularly in developing "operational" indicators, i.e. those which can be used in formulating national or local social programmes.

Use of statistics to estimate the social needs, i.e. the supply and distribution of "social units" required for maintaining or improving the levels of living 3.3

The indicators discussed above reflect the aims of social programmes by defining in statistical terms commonly accepted aspirations in each field. The next question that may have to be answered by the policy maker or programing officer is: what is the magnitude of the job to be done, in terms of dwellings to be built, food to be supplied and distributed, medical and educational services to be made available, by the public and private sectors, in order to maintain or improve prevailing living conditions? The methodological problems involved in estimating social needs have been explored but not settled. It must be mentioned in this connexion that the Twentieth Century Fund Surveys,[10] which include estimates for the United States for 1960, contain an impressive amount of information on this subject.

Social needs must be related either to a point of time (e.g. a certain date), as in the case of estimates of the housing shortage, or to a certain time period, as in the case of food supplies. The volume of work to be done, i.e. the social programmes, however, can only be related to a period of time which may be one year, three, five or other number of years. A housing programme, for example, may be designed to absorb the existing housing shortage during a 5-, 10-, or 20-year period and, consequently, the magnitude of the annual job to

[10] Dewhurst, Frederic J. and Associates, *America's Needs and Resources, A New Survey*, New York: The Twentieth Century Fund, 1955.

be done will change in accordance with the length of the programme. The following discussion is concerned with estimates of needs in terms of goods or services to be provided by the programme over a period of several years.

The possibility of making objective estimates of social needs may be illustrated with reference to the field of housing. In this case the question is: how many dwellings and housing units of other classes[11] would need to be built in a country or major geographic division, during a given period, in order to ensure that the percentage of people living in dwellings remains stable, or increases, and that both the density of occupation (persons per room) and the percentage of dwellings with three or more persons per room remain stable or decrease? Questions of this type have been answered for many countries at various periods; among others, for Australia, Belgium, Chile, Colombia, Denmark, France, Japan, Netherlands, Peru, Sweden, United Kingdom, United States, and Venezuela, and a review of the assumptions and methods used is being prepared in the United Nations Statistical Office. In making such estimates, statistics have been used in connexion with the following aspects of the problem:

a. To estimate the number of dwellings that would be needed to house the population currently living in housing units considered unacceptable from a structural point of view, e.g. huts, caves, "viviendas callampas", etc., or who are without shelter of any kind,
b. To estimate the number of dwellings required to reduce the density of occupation,
c. To estimate the number of dwellings required to eliminate undesirable and involuntary doubling up of families and households,
d. To estimate the number of regular dwellings that need to be repaired or replaced because of their state of maintenance,
e. To estimate the number of dwellings that will need to be replaced because of obsolescence, disaster or conversion to non-residential use, etc.
f. To estimate the number of dwellings required to house the future population increase, and the potential new households, over a period.

Estimates of total needs have been obtained by combining the partial estimates indicated above or by other means.

In the case of housing programmes the essential "goods" is the dwelling, and in this discussion only one type of dwelling has been considered. Other programmes are concerned not with one type of goods or services but with the supply and distribution of several or many goods and services, separately or in alternative combinations. In order to estimate social needs it would be necessary to define which are the essential units in each case. For primary education programmes the essential unit may be the teacher, for health programmes there would be several units and the doctor might be one of them. The determination of which are the basic social units to be supplied or made available to the population in the case of each programme is an essential question which must be settled in order to estimate the volume or number of units needed and whether such needs could be satisfied with the available resources.

[11] The concepts of "dwelling" and "housing unit" used in this discussion are those defined in United Nations, *General Principles for a Housing Census*, Document No. ST/STAT/SER.M/28, New York: United Nations, Sales No. 58.XVII.8.

Use of statistics to test the compatibility of the estimated social needs with the expected economic and human resources 3.4

It would be of little use to estimate the volume of activities necessary to maintain or raise the levels of living without verifying whether such a programme is feasible from the point of view of resources available. Such verification would need to be conducted in respect of financial resources; human resources (such as doctors, engineers or labourers); expected availability of material elements (such as drugs, vaccines, books, foods or building materials); and the capacity of the agencies and industries concerned to undertake their share of the programme. The compatibility of social needs and resources is tested by comparing the estimated needs with the corresponding resources. Information on the following three aspects would be necessary, with respect to each programme, in order to conduct such tests:

a. What are the goods and services, i.e. the "social units" which need to be supplied to the population in order to maintain or raise the levels of living,
b. How many "social units" are needed to produce a certain effect on the statistical indicators of the levels of living,
c. What is the average cost of the "social units" involved in each programme.

Information on these three aspects can be obtained in the case of housing programmes but it may be more difficult to obtain in the case of other programmes. The "social units" would change from country to country according to circumstances, social organization and technical knowledge prevailing at the time the programme is formulated. The problem involved in obtaining the information described in (b) is complicated by the fact that the same changes on certain indicators of the levels of living (e.g. crude death rate) may be obtained by the effect of different "social units", or combinations of them, and that in the social field it is not normally possible to isolate the results of specific actions. However, this complication is essentially peculiar to health programmes whose objectives are vaguely defined by the indicators of the levels of health (see part 3.2). The solution might be facilitated by narrowing down the field of health to aspects directly and primarily related to medical or sanitary actions, or by conceptually sub-dividing the health component of the levels of living into "sub-components".

The testing of the compatibility of social needs and physical resources implies a knowledge of the physical inputs of various kinds that constitute a certain "social unit". In the field of housing for example, it would be necessary to know how much of each class of basic materials, how many hours or months of work of non-skilled, skilled and professional manpower, etc., go into the construction of an average dwelling of a certain type. These inputs when multiplied by the number of dwellings needed to obtain the desired effect on the housing levels of living would yield estimates of physical needs for housing programmes which could then be compared with the available resources.

The capacity to produce or import certain essential elements (for example building materials in the case of housing programmes, vaccines and drugs in the case of health programmes, books in the case of education programmes, basic foods, etc.), as well as the availability of professional and skilled personnel,

are essential limiting factors for social programmes, just as essential as the financial resources. The case of countries which could finance an adequate social programme but do not have enough teachers, doctors, engineers, nurses, or statisticians is not unusual. A much more frequent case is, of course, that in which both financial and physical resources are scarce.

The availability of professional personnel has especial importance for social programmes, as can be illustrated by the following example taken from "The Approach of Operational Research to Planning in India" by P. C. Mahalanobis: "There are, at present, about 65,000 fully qualified (six-year trained) doctors in India. . . . About two thousand six-year trained doctors are turned out in India every year, and the cost of training each doctor is about forty or fifty thousand rupees. Under existing conditions it may take 60 or 70 years to provide one doctor for every two thousand persons in the rural areas on an average . . . I have thought it desirable, therefore, to include in the Draft Plan-frame a proposal to bring some health service to every home in the country within a reasonable time, possibly in 10 or 15 years . . . , by establishing two new cadres of 2-year and 4-year trained health assistants as a first step to a national health service throughout the country. . . . One 6-year trained physician would be in charge of a group of 5 or 6 health assistants; . . . "[12]

The testing of needs and economic resources in monetary terms can be made with respect to the total resources of the country or with respect to the households' income. The use of statistics for testing the consistency of national resources with the magnitude of social programmes is of great importance, since social programmes must be realistic but sufficiently adequate to ensure a gradual improvement of living conditions. The total economic resources needed for a social programme would be determined by relating to the number of "social units" needed their unitary cost. The aggregate cost of a national programme may then be compared with the resources estimated to be available for that programme (see part 3.1).

The following example taken from the field of housing will serve as an illustration: In a report on housing in Peru[13] it is estimated that it would be necessary to construct 2,339,700 dwellings and to repair some 1,011,500 over a 30-year period, beginning on 31 December 1956, in order to improve the housing conditions to certain levels now considered acceptable in that country. This would represent an investment of approximately 60,000 million soles at 1956 prices, as indicated in Table 3.

The total cost of the programme is related to the national income as follows: "The cost of the housing programme, excluding communal services, has been estimated to be 60,000 million soles, to be invested in 30 years. Since the national income and, parallel with it, the capacity for dwelling construction will increase during this period, the most convenient form of distributing such investment over the 30-year period is by equating it to a fixed proportion of the national income.

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[12] Mahalanobis, P. C., "The Approach of Operational Research to Planning in India," Reprinted from *Sankhyā, The Indian Journal of Statistics*, Vol. 16, Parts 1 and 2, Chapter 5, para. 7, Calcutta: Statistical Publishing Society, 1955.

[13] Comisión para la Reforma Agraria y la Vivienda, *Informe sobre la vivienda en el Perú*, Lima, Perú: Imprenta Casa Nacional de Moneda, 1958.

TABLE 3. COST OF 30-YEAR HOUSING PROGRAMME, BEGINNING 31 DECEMBER 1956

Areas	New dwellings to be constructed			Existing dwellings to be repaired		
	Number of dwellings	Cost per dwelling (Soles 1956)	Total investment (Soles 1956)	Number of dwellings	Cost per dwelling (Soles 1956)	Total cost of repairs (Soles 1956)
			(millions)			(millions)
Metropolitan	657,200	40,000	26,288	68,100	10,000	681.0
Urban	669,400	30,000	20,082	255,800	7,500	1,918.5
Rural	1,013,100	10,000	10,131	687,600	2,000	1,375.2
Total	2,339,700		56,501	1,011,500		3,947.7

Source: Comisión para la Reforma Agraria y la Vivienda, *Informe sobre la vivienda en el Perú*, Lima, Perú: Imprenta Casa Nacional de Moneda, 1958.

"On this basis and on the assumption that both the population and the national income will increase following a geometric progression, with annual rates of 2.1 and 5 per cent, respectively, the annual investment can be estimated.

"The investment for the first year would be 900 million soles, and this amount would be increased by 5 per cent per year, that is, in the same proportion that the national income increases in constant prices. In this form 60,000 million soles would be invested over the 30-year period.

"Considering that the national income in 1957 was of the order of 25,000 million soles . . . and that the available proportion for net investment was 16 per cent (average for the last five years), the resources for investment would be 4,000 million soles. The 900 million investment needed for the first year of the housing programme represent approximately 3.6 per cent of the national income and 22.5 per cent of the net investment, which are evidently very high percentages."[14]

The use of statistics to determine the extent to which households are able to finance their essential social needs out of their regular income is quite common. The question to be answered may be proposed in the following form: Given a certain average level of income, a distribution of households by income groups and corresponding patterns of expenditure, what proportion of the households could, or could not, afford to purchase the "social units" considered in social programmes (housing, food, education, medical services, etc.) at current or expected prices? For example, the question of whether households can afford a type of minimum satisfactory dwelling is normally explored in the formulation of national housing programmes.

In a fairly recent study on the economic significance of housing in Central America and Panama, it is estimated that approximately 46 per cent of the urban families in El Salvador could not pay more than 70–88 per cent of a monthly rent equivalent to 1/100 of the average cost of a dwelling, such rent

[14] *Ibid.*, pp. 57–8 (Unofficial translation).

being considered adequate to finance the dwelling construction, ("economic rent").[15] In another study dealing with the housing situation in Colombia, about 20 per cent of urban families were found unable to pay a rent equivalent to the amount needed to finance the construction of a minimum acceptable dwelling in forty years.[16]

The form in which the information on household income and expenditure may be used in the formulation of social programmes deserves separate consideration, as, indeed, do all the other major uses of statistics briefly examined here. It is not possible to go any further into this matter in this discussion.

Use of statistics to select the most convenient combination of activities, i.e. a social programme 3.5

After social needs are estimated and matched with the estimates of available resources, the next logical step would be to attempt to bring needs and resources into balance by adopting feasible targets or goals for social programmes. Such targets could not be entirely objective since they would be based on assumptions regarding future population growth,[17] national income and relationships between social measures and levels of living, and would be the result of compromise between desire for improvement of one aspect of the levels of living (e.g. housing), the need to maintain an even rate of improvement in other components and the need to maintain a sustained economic growth. The targets, furthermore, should be revised periodically as new information becomes available. The statistical bases on which the targets would be established would be known and could be examined by anybody interested in the subject, a fact that should have considerable political significance, especially in under-developed countries.

The resources may be found to be smaller or larger than the needs for a certain social programme. Normally the balance of needs and resources for any programme will vary depending on the type of resources. There may be sufficient labour and insufficient building materials, sufficient financial resources and insufficient professional personnel, and so forth. Whatever the situation may be, the use of statistics in exploring the possibilities of equating needs and resources would be essential.

If after exploring all the possibilities, the resources are insufficient even to maintain the existing levels of living, statistics can also be used to evaluate the consequences of such a situation. In the case of housing, to mention a very real example, statistics could be used to estimate the extent which the housing shortage would increase in the event that an insufficient number of dwellings were built over a period of years. Statistics, however, would not be very helpful in judging the political and social consequences of allowing housing conditions to deteriorate consistently over a period of several years.

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[15] United Nations, Economic Commission for Latin America, *Significación económica de los programas de vivienda en Centro América y Panamá*, Document No. E/CN.12/CCE/AC.6/5, New York: United Nations, 1 November 1957.

[16] Corporación Nacional de Servicios Públicos, Departamento de Vivienda, *Déficit y demanda de vivienda en Colombia*, Series Estudios Socio-Económicos No. 3, Bogotá, Colombia, 1956.

[17] Implied in projections of national income as well as in the estimates of future needs.

4. USE OF STATISTICS FOR THE EVALUATION OF SOCIAL PROGRAMMES

While the formulation of a social programme is an *a priori* operation, oriented to the future, the evaluation must be made *a posteriori*, and should, therefore, be based on past experience.

In this discussion a social programme has been considered to be composed of public and private activities, whether or not consciously oriented. When the programme has been established by a systematic procedure, such as the one broadly outlined in part 3 of this paper, it could be evaluated in the following ways:

a. By measuring the ultimate results of the programme by the observed changes in the indicators of the levels of living,
b. By testing the assumptions made in formulating the programme as regards the expected behaviour of the households, future population growth national income and expected results of social measures in the light of observed facts,
c. By assessing the administrative efficiency of the agencies concerned.

The measurement of levels of living by means of statistical indicators has been discussed in part 3.2 and the evaluation of assumptions, "laws" and theories or of the administrative efficiency of agencies cannot be approached in a general way. Such evaluations can only be made with reference to the particular assumptions, laws etc. related to individual programmes. In the final analysis, the only comprehensive yardsticks of the success or failure of social programmes are the statistical indicators of the levels of living.

CONSUMERS' PROPENSITIES TO HOLD LIQUID ASSETS*

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There is wide variation in the need for and the ability to accumulate liquid assets. The model used for investigating liquid asset holdings attempts to isolate the precautionary motive from other motives for holding liquid reserves. The propensities for holding liquid assets for precautionary purposes are analyzed by classifying consumers according to their long-run income position. Within each class regression equations are used to test hypotheses concerning the relationship between liquid balances and age, size of consumer unit, level of current income, and home ownership.

The results suggest that the propensity to hold liquid assets (1) is low for consumers who can expect long-run increases in income, (2) is higher for consumers who have permanently low incomes, but that these consumers may feel that their reserves are inadequate, (3) increases with age to a limit, and (4) decreases as the consumer unit becomes larger.

THE effect of liquid asset holdings on consumers' saving, once regarded as a persistent and simple negative relationship, has not been clearly demonstrated in empirical tests. The results of some tests using cross-section data have suggested that the asset effect is variable over time and that it may have both positive and negative signs. Studies by Klein [4, pp. 232–8] and Tobin[1] indicate that although liquid asset holdings had a relatively strong positive effect on durable goods expenditures in 1948, the effect was weaker in 1949 and had disappeared by 1951–2. Katona [3, pp. 167–70] has suggested that there are two different asset effects. Some consumers with high asset holdings may feel relatively saturated with assets and therefore save less than those with low asset holdings. Other consumers with high asset holdings may have formed a habit of saving, or may be more thrifty, and therefore save more than those with low asset holdings. Tobin and Morgan [7, pp. 124–6] found that the sign of the asset effect depends on income change. According to Morgan, saving was positively associated with liquid asset holdings when the consumer's income had increased substantially and negatively associated with liquid asset holdings when his income had declined substantially.

The variety of hypotheses implicit in these findings suggests the need for an examination of liquid asset holding by consumers. Why do liquid asset holdings differ between consumer units? Why do they vary over time for given consumer units? Knowledge of the needs fulfilled by liquid asset balances seems to be a logical basis for hypothesizing the effect of the balances on saving. The following sections describe the results of an attempt to discover why some consumers

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[1] James Tobin has kindly permitted reference to results of research not yet published.

hold larger liquid asset balances than others and to explain the differences in levels of liquid assset holdings among specified kinds of consumers over time.[2]

1. THE MEASUREMENT OF LIQUID ASSET HOLDINGS

Consider two consumers, A and B, each holding \$1,000 in liquid assets. Suppose that A has an annual income of \$1,000 while B has an annual income of \$10,000. Although each has saved the same absolute amount it is clear that the balances have been accumulated under different circumstances. The wide difference in incomes indicates that the two consumers probably have quite different standards of living. Furthermore it is reasonable to assume that A and B differ in their perceptions of need for liquid assets.

The two consumers can be distinguished by expressing their liquid balances as a per cent of annual income. The ratio of liquid assets to income for B is 10 per cent; the ratio of liquid assets to income for A 100 per cent. Thus the ratio of liquid asset holdings to annual income is a more meaningful measure than absolute balances for an analysis of liquid asset holdings. Hereafter it will be referred to as the LA ratio.

The LA ratio has been used as an explanatory variable in several studies of saving functions. The LA ratio is chosen over the absolute level of liquid asset holdings in saving functions primarily on grounds of partial elimination of the positive relationship between asset holdings and income, and the reduction of variance in saving by dividing both sides of the saving function by income. Both of these reasons for dividing liquid asset holdings by income are relevant to the present analysis, but perhaps even more important is the need for a measure of asset holdings which is appropriate for an investigation of the consumer's rationale for holding liquid assets. Assuming that the consumer's desire to hold liquid assets is related to his general standard of living, which is approximated by his income in a given year, the LA ratio measures liquid assets in an appropriate context.

The data analyzed were gathered in Surveys of Consumer Finances by the Survey Research Center, University of Michigan, on behalf of the Board of Governors, Federal Reserve System. Liquid assets are defined as assets of fixed money value which can be converted to money quickly and conveniently. Liquid assets, as measured in the surveys, consist of holdings at the end of a given year in the form of savings deposits in banks and other savings institutions, checking accounts, and U. S. bonds. Currency and the cash surrender value of life insurance policies, although liquid in the economic sense, are not included in the surveys for technical reasons. Total income includes cash income received by all members of the spending unit.

Each interview is assigned a numerical weight to correct the data for variable sampling rates between income groups and for differences in response rates. These weights have been used in all calculations presented here.

2. THE RATIONALE OF HOLDING LIQUID ASSETS

The usual motivations for holding money can also be ascribed to the holding of liquid assets. Spending units hold liquid assets for transaction purposes, for

[2] The important question of aggregation, the relationship of the various explanations discovered through micro-analysis to the aggregates of liquid asset holdings, is not considered here.

speculative purposes, or for precautionary purposes, and all three motivations might apply for many families. This analysis of *LA* ratios among spending units cannot distinguish clearly between these motivations but it can lead to some general conclusions about the relative importance of the precautionary motive between different kinds of spending units.

Except for differences in the propensity to consume between consumers, differences in the *LA* ratio should not show the effect of the transaction motive. Two consumers who receive different monthly salaries, each consuming his total salary from a checking account at a constant rate will each have an average *LA* ratio of 1/24 over the month. One month can be considered the longest period for which wage payments are made in practice. Moreover, there may be economies of scale in transactions for the higher income families which would result in transaction ratios of less than 1/24. Therefore, 1/24 approximates a maximum *LA* ratio for transaction purposes. The average *LA* ratios observed for many spending units in the Surveys of Consumer Finances are much larger than 1/24, even when only checking account balances are used in the numerator. The empirical analysis therefore assumes that observed differences in *LA* ratios cannot be attributed to the transactions motive.

It is more difficult to eliminate the effect on the *LA* ratio of assets held for intended consumption or intended investment in securities or commodities. At a given point of time one consumer may hold $1,000 for unforeseen contingencies. Another may hold the same amount for an intended purchase of a house. It is not possible to distinguish between the two motives by observation of the consumers' *LA* ratios.

The influence of assets held for intended consumption or investment can be partially identified, however, by comparisons over time. The empirical results reported here compare *LA* ratios for specified types of spending units in 1947 with the *LA* ratios for similar—but not identical—spending units in 1953. During the intervening period the supply of consumer goods increased rapidly, there was a boom in real estate and common stock, and interest rates on liquid assets were held at low levels through most of the period. It is reasonable to assume that assets held for intended consumption or investment had little influence on *LA* ratios in 1953 compared to their influence in 1947. The forced saving during World War II resulted in a disequilibrium with respect to liquid asset holdings in 1947 relative to 1953.

Still another type of motive for holding liquid assets must be considered, a type that is less rationalistic than the others, but one that is subjectively important to the holders of the assets. There are many people who hold liquid assets because they are unaware of investments bearing a higher yield, because they are averse to any risk of money value, or simply because they do not think of non-liquid assets as being marketable. The last point seems to apply to a substantial number of respondents who, in a recent study conducted by the Yale Workshop in Quantitative Economic Research, were asked, "What is the best kind of asset to pass along (in an estate) to your children?" Many of the replies mentioned various forms of liquid assets. The reason for this choice was typically that money is easily divided and family quarrels are avoided. These motives must be acknowledged but we shall consider them to be subsumed under the precautionary motive, the desire to have money conveniently

at hand when emergencies arise. If emergencies do not arise the assets fulfill a residual function in yielding a small but sure return and as a store of money value for consumption during retirement or for bequests.

The empirical analysis of the relative strength of the precautionary motive among certain groups of consumers is based upon the following postulates:

1. During a given year differences in the L*A* ratio between groups of consumers cannot be attributed to the transactions motive.
2. Because of forced saving during World War II the year 1947 is "abnormal" with respect to liquid asset holdings. Differences in the L*A* ratio between groups of consumers arise from differences in the precautionary motive and from differences in intended expenditure.
3. The year 1953 is "normal," relative to 1947, with respect to liquid asset holdings, and differences in the desires for precautionary reserves account for large differences in the L*A* ratio.
4. Differences in the L*A* ratio between 1947 and 1953 for similar consumer groups indicate whether or not the L*A* ratios of 1947 were "normal"—and therefore held primarily for precautionary purposes—or "abnormal"—and therefore held for both precautionary purposes and intended expenditure.

3. THE HYPOTHESES TESTED

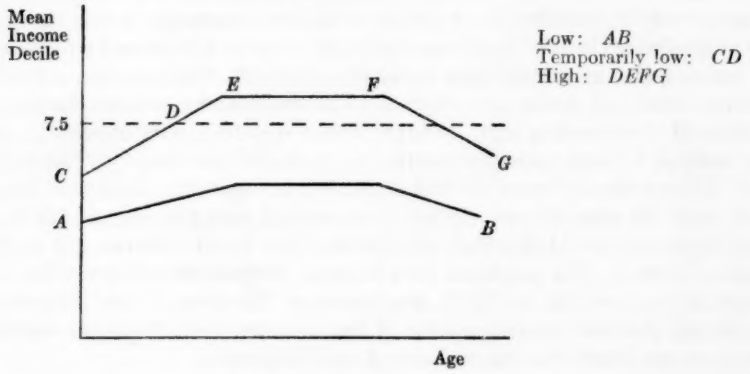
The hypotheses were developed after examining L*A* ratios for 1947 and 1951 [2]; the tests have been performed on data for 1947 and 1953.[3] The formulation of the hypotheses depended heavily on a consideration of the outcome of the conflict between the need and desire for consumption versus the need and desire for precautionary reserves. The outcome is expected to differ according to the number of persons in the spending unit and the position which the spending unit occupies in a profile of income over the life span of the head of the spending unit.

The preliminary examination of the L*A* ratios in 1947 and 1951 suggested that single-person spending units were so markedly different from multiple-person spending units that they should not be combined in a hypothesized continuum. Family formation apparently results in pressures to consume which lead to a lower propensity to hold liquid assets. Single-person spending units have therefore been separated from multiple-person spending units for empirical analysis.

The preliminary examination of L*A* ratios also suggested that the propensity to hold liquid assets varies with income in very complex relationships. To account for the income effect in the present study three separate income classes are defined by the occupation, education, age, race, and regional location of spending units. The latter variables define groups of spending units and the mean income decile has been computed for each group.

[3] Data for the latter years are contained in the 1948 and 1954 Surveys of Consumer Finances, respectively. The data for 1947 were used both to develop and to test the hypotheses. Although this is not good statistical practice, data for 1946 were not fully comparable with data for subsequent years, and by 1948 consumers were well advanced in the process of adjusting to the postwar disequilibrium. Statistical purity was therefore sacrificed for a sharper economic effect.

For example, one group consists of white spending units living in the South, the heads of which were skilled workers with a high-school education and were between 35 and 44 years old. For each spending unit in the group there is an income decile position, from 1 (low) to 10 (high), which is determined by the relative position of his absolute income in an array of incomes for all spending units. The weighted mean of the individual income deciles within a group is the basis for combining groups of spending units into income classes. The income decile is used for classification rather than absolute income because relative income is assumed to be more stable over time. This comparative stability is also important in the regression model explained below. The groups were combined into classes by examining the mean income decile over the age span for each race-region-occupation-education group. If the mean decile is less than 7.5 for all ages the entire race-region-education group is classed as "Low." If the mean income decile is less than 7.5 for the younger age groups but 7.5 or more for at least one of the older age groups, the younger groups are classed as "Temporarily Low." The groups which have mean income deciles of 7.5 or more, or which are older than such a group, are classed as "High." The following diagram illustrates the classification procedure:



The income classes are intended to reflect the spending units' income expectations. The "High" class includes spending units who can expect to maintain a high relative income position. The "Low" class is composed of spending units whose status is such that they could not expect to rise above a decile position of 7.5. The "Temporarily Low" class consists of young people whose current relative income position is similar to the position of those in the "Low" class, but who can expect to rise to the upper levels of relative income as they grow older. A table showing the mean income decile position for all race-region-occupation-education-age groups is shown as an Appendix.

It is possible that spending units who have different income expectations will not react in the same way to given changes in current income. In order to measure differences in the effect of current income on the LA ratio between income groups, the observed value of income decile is used as an independent

variable in regression analyses for each expected income group. The use of income decile rather than measures of absolute income allows direct comparison of income effects between 1947 and 1953.

There are strong *a priori* grounds for expecting consumers to hold more liquid assets for precautionary purposes as they grow older. First, the need for such reserves increases with age because the threat of adverse economic events (e.g., illness, disability, loss of favorable employment opportunities) is greater for older people than for younger people. Second, the positive relationship between income and age gives older people the ability to accumulate liquid reserves. Finally, the preference for holding liquid assets vs. consumption may be expected to become stronger as the pressures for consumption decline with age. On all three points--need, ability, and preference—precautionary reserves should increase with age. The results of the tests are consistent with this hypothesis.

As the size of a spending unit grows larger, the need for precautionary balances should increase. An adequate reserve for medical expenses and other contingencies for families of several persons may be expected to be greater than the reserve needed for smaller families. The pressure for current consumption also increases with family size, however, and the preference which families express between the satisfaction of desires for current consumption and the security of the desired level of liquid reserves is not obvious. Let us posit a preference for consumption as the stronger of the two demands. There is then a conflict between need and preference which is to be resolved by ability, the income position of the spending unit. As high income spending units increase in size, they are able to fulfill their consumption demands and also maintain the desired level of precautionary reserves. But as low income spending units grow larger, their needs for current consumption have priority over the demand for liquid reserves. It is expected that they will increase their liquid balances only as their incomes increase. The empirical tests support the posited preference for consumption over security as family size increases. The data do not support the hypothesis that the income position of the spending unit affects the relationship between liquid reserves and size of spending unit.

Although this analysis concentrates on liquid assets as one of the more important forms of wealth held by consumers, the size of holdings in other forms ought to affect liquid balances. The high frequency of life insurance ownership seems to indicate that life insurance is regarded as a favorable substitute for other forms of wealth. The available data permitted testing only the simple hypothesis that precautionary balances are negatively related to the spending unit's equity in an owner-occupied home. Since it is reasonable to assume that home ownership satisfies a desire for security this hypothesis tests for one relevant kind of substitution effect. The results of the test indicate that there is not a general substitution effect between home ownership and liquid balances.

4. THE DATA AND THE REGRESSION MODEL

The original samples drawn for Surveys of Consumer Finances represent the entire consumer sector with minor exceptions, e.g., institutional and transient population. Several segments of the original samples have been omitted from

this analysis. Farmers and owners of unincorporated businesses have been omitted because it is difficult, and in some cases impossible, to separate business assets from household assets. Although non-white spending units had LA ratios which were significantly lower than LA ratios among white spending units, all hypotheses attempting to explain differences in LA ratios among non-whites were rejected, and these spending units were excluded from further analysis. Similarly, spending units in which the head is retired have LA ratios significantly higher than the LA ratios of other spending units, but hypotheses formulated to explain differences in the LA ratios had to be rejected. Retired spending units were therefore omitted. A small number of spending units have LA ratios which, because their incomes are very low, are extremely high relative to others. To avoid the distorting influence of these extreme values, LA ratios of 1,000 per cent or more have been omitted. It was also necessary to omit those spending units for which values of one or more of the independent variables were not ascertained.

After these omissions the remaining observations were classified as follows according to the procedures described above:

Class	Number of cases	
	1947	1953
Single-person spending units:		
High income	86	100
Temporarily low income	78	79
Low income	313	266
Multiple-person spending units:		
High income	520	584
Temporarily low income	188	231
Low income	941	678
Total	2,126	1,938

For each of the six classes and for 1947 and 1953 separately a regression equation of the following form has been calculated:

$$100\frac{LA}{Y} = b_0 + b_1A + b_2N + b_3D + b_4D^2 + b_5(OA) + u$$

where:

100 LA/Y = the LA ratio described above, expressed in per cent

A = age of head of spending unit in the scale:

1 = age 18–24
2 = age 25–34
3 = age 35–44
4 = age 45–54
5 = age 55–64
6 = age 65 and over

N = number of persons in the spending unit except that $N = 0$ for single-person spending units

D = value of income decile for the individual spending unit regardless of its income class

D^2 = squared value of D

O = home ownership = 1 if owns home
0 if does not own home

u = residual

The curvilinear form for the influence of D on the LA ratio fits the U-curve of LA/Y plotted against D for all spending units. The OA variable is a proxy for equity in an owner-occupied home assuming that equity is positively related to the age of the owner.

5. ANALYSIS OF THE LEVEL OF THE LA RATIO IN 1953

Table 1 shows the mean value of the LA ratio and the regression coefficients in 1953 for each group. Table 2 shows an analysis of variance of the level of the LA ratio within classes with the regressions not taken into account. Table 3 shows the F ratios which test for significant net differences in regression coefficients between classes, accounting for the joint effect of all regression variables and coefficients. Table 3 tests the hypothesis that the variance explained by separating classes is equal to the remaining variance of u after separation of classes. Both Tables 2 and 3 test for differences between single- and multiple-person spending units, and between the three income classes.

TABLE 1. VALUES OF REGRESSION COEFFICIENTS AND MEANS OF THE LA RATIO IN 1953 FOR SPECIFIED CLASSES
(standard errors shown in parentheses)

Class	Mean LA Ratio	Regression Coefficients					
		Intercept	Age	Number	Decile	Decile squared	Home Ownership
Single-person Spending Units:							
High income	56.53	19.53 (51.78)	22.63* (8.47)	—	−15.04 (18.85)	1.35 (1.42)	−11.38* (4.91)
Temporarily low income	25.48	63.91* (22.33)	4.10 (9.94)	—	−21.83* (10.69)	2.19 (1.36)	[1]
Low income	62.98	54.73 (32.52)	13.66* (4.65)	—	−31.31* (14.05)	3.83* (1.67)	1.57 (3.58)
Multiple-person Spending Units:							
High income	33.20	37.23 (28.71)	10.41* (2.65)	−6.76* (1.75)	− 9.90 (7.56)	.93 (.53)	.57 (1.34)
Temporarily low income	9.33	4.99 (8.38)	.08 (2.15)	−3.06* (.86)	4.15 (2.57)	− .26 (.20)	− .25 (1.05)
Low income	29.98	28.56 (18.68)	11.38* (2.44)	−6.09* (1.85)	− 7.05 (5.60)	.48 (.47)	2.64 (1.54)

[1] Not calculated because there were no home owners in the class in 1947. Consistent procedures were used in both years to allow comparisons between years.

* Regression coefficient significantly different from zero at a level less than 5 per cent.

TABLE 2. ANALYSIS OF VARIANCE FOR DIFFERENCES BETWEEN LEVELS OF THE *LA* RATIO IN 1953

Source of Variation	Sum of Squares		Degrees of Freedom		Estimated Variance		*F*
	Explained	Unexplained	Used	Remaining	Explained	Unexplained	
Spending Unit	277,566	11,823,179	1	1,936	277,566	6,107	45.45**
Income	178,425	11,922,319	2	1,935	89,212	6,161	14.48**

** Significant difference at the 1 per cent level.

The mean *LA* ratio is higher among single-person spending units than among multiple-person spending units. Moreover, the regression coefficients indicate that a reduction of the *LA* ratio is associated with each additional person in the spending unit. A tentative conclusion based on 1953 data is that as family size increases, consumption (and other forms of saving) gains in preference over holding liquid assets.

The significant difference between income classes shown in Table 2 is accounted for, in part, by the low *LA* ratio for the temporarily low income group. By definition this group contains relatively young spending units, however. The regression coefficients indicate that the *LA* ratio is positively related to age in both high and low income groups. The low *LA* ratio among spending units with temporarily low incomes is probably explained in part by their age.

According to Table 3 there are no significant net differences in the regression coefficients between classes when all coefficients are tested jointly. Inspection of the individual coefficients in Table 1 suggests, however, that the effect of the income decile position is different between classes. The income coefficients in the multiple-person spending unit classes are not significantly different from zero. Among multiple-person spending units, regardless of income class, the *LA* ratio is not affected by income differences. Similarly in the high income, single-person spending unit class, the *LA* ratio is not affected by income decile position. In both low income, single-person spending unit classes there is an income effect, however. Those who have temporarily low incomes show declining *LA* ratios as their income position is improved. Spending units who, by their

TABLE 3. *F* TESTS FOR DIFFERENCES IN REGRESSION COEFFICIENTS BETWEEN GROUPS, 1953

Groups Tested	Sums of squares of *u*			Degrees of Freedom				*F*
	Unexplained		Explained	Used			Remaining	
	Pooled	Separate		Pooled	Separate	Net		
Spending Units	10,711,883	10,673,744	38,139	6	11	5	1,926	1.38
Income	10,711,883	10,668,707	43,176	6	18	12	1,919	.65

occupation-education characteristics, have been classed in a permanently low income group show changing preferences for liquid assets as their income increases. In the lower range of income deciles the *LA* ratio is reduced as income increases, but with further increases in income the *LA* ratio rises.

In only one class, the high income, single-person spending units, is there any evidence that equity in a home substitutes for liquid asset holdings. The variable for measuring equity is crude, and it yields no information about the magnitude of the substitution effect. Nevertheless, it is interesting to note the occurrence of the substitution effect only in the class in which the need for liquid asset holdings is relatively low and the ability to save is relatively high. The data suggest that, except for the very minor group noted above, consumers do not shift between asset forms while maintaining some normative level of security in total wealth. Rather it is possible that each form of asset has a desired normal level and there is no "portfolio problem."

The analysis of the level of the *LA* ratio in 1953 suggests the following tentative conclusions. Once a household of more than one person is established the *LA* ratio is determined by the age of the head and the size of his family. Furthermore, there are no economically significant differences in the effect of these variables between the high and low income classes. The propensity to hold liquid assets among single-person spending units varies with their income class, however. Those in the high income class have a relatively high average *LA* ratio but individual *LA* ratios are not associated with differences in the income decile. There is some evidence of satisfaction with the level of the *LA* ratio in this group in the tendency to let equity in a home substitute for higher *LA* ratios. Those who can expect high income, the temporarily low income class, have a low propensity to hold liquid assets. But additional income is evidently diverted primarily to consumption and, possibly, other forms of saving, indicating satisfaction with the low *LA* ratio. The single-person spending units who have low incomes and cannot expect to rise to high income positions show a latent unsatisfied demand for higher liquid balances. Their average *LA* ratio is comparable to the average *LA* ratio in the high income class but it tends to increase in the higher range of income deciles.[4]

6. ANALYSIS OF DIFFERENCES IN *LA* RATIOS BETWEEN 1947 AND 1953

A dynamic analysis of the change in the *LA* ratio over time for identical spending units—if it could be performed—might reveal clearly the motivations for holding liquid balances. The data available do not describe identical spending units in two years. If we regard the motivations for holding liquid assets as a set of preferences between security and current consumption—the preferences varying with income expectations and family structure—we may expect to find differences in behavior over time between *similar* spending units. We may expect that different kinds of spending units reacted to the relatively large liquid balances accumulated during World War II according to differing preferences. It is possible that the forced saving of the early 1940's produced abnormally large liquid balances for some spending units. To other spending units the war-

[4] An alternative interpretation is that only those few persons who attain higher income levels are concerned. As a result of a desire for higher liquid balances they may work harder than others, work overtime, or hold more than one job in an effort to raise their *LA* ratio.

TABLE 4. MEAN VALUES OF THE *LA* RATIO FOR SPECIFIED CLASSES OF SPENDING UNITS, 1947 AND 1953

Class	1947	1953	% Change
Spending Units			
$N=1$	53.9	55.2	+ 2
$N>1$	37.1	27.8	−25
*Income Groups**			
High	54.1	36.6	−32
Temporarily Low	25.3	13.2	−48
Permanently Low	39.7	39.0	− 2
Spending Unit and Income			
$N=1$ High	92.7	56.5	−39
$N=1$ Temporarily Low	28.8	25.5	−12
$N=1$ Permanently Low	51.3	63.0	+23
$N>1$ High	46.9	33.2	−29
$N>1$ Temporarily Low	23.7	9.3	−61
$N>1$ Permanently Low	35.4	30.0	−15
*All Spending Units**	41.2	34.0	−17

* Differences in mean *LA* ratios between years are statistically significant; see Table 5.

time accumulations may not have seemed abnormal even though they had never held such large liquid balances previously. Given that the consumer sector held unprecedented liquid balances at the end of World War II, for which kinds of consumers were these balances abnormal in terms of their motivations to hold liquid assets? One kind of answer is found by comparing the levels of the *LA* ratios between 1947 and 1953 for different classes of spending units.

Table 4 shows the mean *LA* ratios in the two years for specified classes of spending units. Table 5 shows the results of an analysis of variance of the *LA*

TABLE 5. ANALYSIS OF VARIANCE FOR DIFFERENCES BETWEEN LEVELS OF THE *LA* RATIO, 1947 AND 1953

Source of Variation	Sums of Squares		Degrees of Freedom		Estimated Variance		*F*
	Explained	Unexplained	Used	Remaining	Explained	Unexplained	
Years (*T*)	51,900	27,024,006	1	4,062	51,900	6,653	7.80**
Spending Unit (*SU*)	366,015	26,709,892	1	4,062	366,015	6,576	55.66**
Income (*Y*)	272,772	26,803,134	2	4,061	136,386	6,600	20.66**
$T \times SU$	15,263	26,642,728	1	4,060	15,263	6,562	2.33
$T \times Y$	50,826	26,700,410	2	4,058	25,413	6,580	3.86*

* Significant difference at the 5 per cent level.
** Significant difference at the 1 per cent level.

ratio for different groups of spending units when time is introduced as an independent variable. Significant differences in mean L*A* ratios between 1947 and 1953 occur for all spending units (*T*), and for spending units classified by income (*T*×*Y*).[5]

Thus there is a significant decline in the L*A* ratio for the entire consumer sector, but the high income class and the temporarily low income class account for the decline. The trivial change in the L*A* ratio among permanently low income spending units suggests that their 1947 liquid balances were not abnormal. The needs and preferences for liquid assets were evidently stronger among these people than among those who had, or could expect, high incomes. If these results evolved from an analysis of identical spending units we would conclude that permanently low income people maintained their post-war liquid reserve position more tenaciously than others. Given similar—but not identical—spending units in the two years, the differences indicate only that the motivations for holding liquid assets at the 1947 level were stronger in 1947 among low income groups than among high income groups.

Table 6 shows the results of *F* tests for significant net differences in the regression coefficients between years, accounting for the joint effect of all regression variables and coefficients. Table 6 tests the hypothesis that the variance explained by separate regressions for the two years is equal to the remaining variance of *u* after separation of the years. The significant net differences be-

TABLE 6. *F* TESTS FOR DIFFERENCES IN REGRESSION COEFFICIENTS BETWEEN YEARS

Groups Tested	Sums of Squares			Degrees of Freedom Used				*F*
	Unexplained		Explained by Separate Years	Years Pooled	Years Separate	Net	Remaining	
	Years Pooled	Years Separate						
All Units	24,000,614	23,876,801	123,813	6	12	6	4,051	3.50**
Spending Units								
$N=1$	9,897,458	9,777,415	120,044	5	10	5	911	2.24*
$N>1$	13,884,898	13,823,726	61,172	6	12	6	3,129	2.31*
Income								
High	4,664,121	4,524,978	139,143	6	12	6	1,277	6.54**
Temporarily low	589,995	563,787	26,208	6	12	6	563	4.36**
Permanently low	18,635,921	18,601,222	34,699	6	12	6	2,185	.68
Spending Unit and Income Groups								
$N=1$ High	1,534,425	1,367,453	166,972	5	10	5	175	4.27**
$N=1$ Temp. low	266,820	255,316	11,504	4	8	4	148	1.67
$N=1$ Perm. low	7,978,182	7,925,230	52,952	5	10	5	568	.76
$N>1$ High	3,032,355	3,006,661	25,694	6	12	6	1,091	1.55
$N>1$ Temp. low	318,785	297,401	21,384	6	12	6	406	4.87**
$N>1$ Perm. low	10,477,974	10,440,015	37,959	6	12	6	1,606	.97

* Significant difference at the 5% level.
** Significant difference at the 1% level.

[5] The test of significance of the interaction effect (*T*×*Y*) requires equal or proportional cell frequencies. The frequencies for the two years are tabulated in Section 4. The numbers of cases in each income group in the two years are assumed to be sufficiently similar, and large, to justify the test.

tween the regression coefficients in the two years do not necessarily imply significant differences between years for particular coefficients. Differences between years for particular coefficients can be submitted to tests, however, in a search for varying effects of the independent variables. Table 7 shows the re-

TABLE 7. REGRESSION COEFFICIENTS AND STANDARD ERRORS FOR SPECIFIED CLASSES, 1947 AND 1953

		Regression Variables					
		Intercept	Age	Number	Decile	Decile squared	Home Ownership
All Spending Units	b_i^{53}	33.12*	12.96*	−6.09*	−10.93*	.92*	.77
	$\sigma_{b_i^{53}}$	(8.99)	(1.37)	(.95)	(2.97)	(.25)	(.96)
	b_i^{47}	34.86*	18.11*	−7.12*	−14.36*	1.37*	− 1.81
	$\sigma_{b_i^{47}}$	(9.66)	(2.18)	(1.03)	(3.13)	(.27)	(1.75)
	$b_i^{53}-b_i^{47}$	− 1.74	− 5.15*	1.03	3.43	− .45	2.58
	$\sigma(b_i^{53}-b_i^{47})$	(13.19)	(2.58)	(1.40)	(4.31)	(.36)	(1.99)
Single-Person Spending Units	b_i^{53}	42.45*	15.09*	—	−20.10*	2.02*	− 1.58
	$\sigma_{b_i^{53}}$	(20.87)	(3.20)		(8.58)	(.92)	(2.68)
	b_i^{47}	5.05	24.24*	—	−14.23*	2.00*	− 1.82
	$\sigma_{b_i^{47}}$	(20.34)	(3.93)		(8.19)	(.84)	(3.13)
	$b_i^{53}-b_i^{47}$	37.40	− 9.15	—	− 5.87	.02	.24
	$\sigma(b_i^{53}-b_i^{47})$	(29.14)	(5.07)		(11.87)	(1.25)	(4.12)
Multiple-Person Spending Units	b_i^{53}	31.53*	11.31*	−6.12*	− 8.69*	.72*	1.70
	$\sigma_{b_i^{53}}$	(11.93)	(1.47)	(1.11)	(3.45)	(.27)	(.95)
	b_i^{47}	76.02*	11.33*	−7.45*	−21.58*	1.82*	.44
	$\sigma_{b_i^{47}}$	(13.05)	(2.88)	(1.30)	(3.59)	(.29)	(2.40)
	$b_i^{53}-b_i^{47}$	−44.49*	− .02	1.33	12.89*	−1.10*	1.26
	$\sigma(b_i^{53}-b_i^{47})$	(17.68)	(3.23)	(1.71)	(4.98)	(.40)	(2.58)
High Income Spending Units	b_i^{53}	39.13	12.17*	−6.19*	−11.69*	1.05*	− 1.34
	$\sigma_{b_i^{53}}$	(21.70)	(2.61)	(1.44)	(5.97)	(.44)	(1.36)
	b_i^{47}	97.09*	22.60*	−9.96*	−30.65*	2.54*	− 6.60*
	$\sigma_{b_i^{47}}$	(26.52)	(4.24)	(1.78)	(7.00)	(.51)	(2.98)
	$b_i^{53}-b_i^{47}$	−57.96	−10.43*	3.77	18.96*	−1.49*	5.26
	$\sigma(b_i^{53}-b_i^{47})$	(34.26)	(4.98)	(2.15)	(9.19)	(.67)	(3.28)
Temporarily Low Income Spending Units	b_i^{53}	37.13*	.89	−3.66*	− 6.03*	.49*	− .46
	$\sigma_{b_i^{53}}$	(7.47)	(2.90)	(.92)	(2.73)	(.23)	(1.60)
	b_i^{47}	6.24	3.32	−2.52	5.35	− .22	− 4.15
	$\sigma_{b_i^{47}}$	(14.97)	(5.40)	(2.14)	(5.23)	(.44)	(3.99)
	$b_i^{53}-b_i^{47}$	30.89	− 2.43	−1.14	−11.38	.71	3.69
	$\sigma(b_i^{53}-b_i^{47})$	(16.73)	(6.13)	(2.33)	(5.91)	(.49)	(4.30)
Permanently Low Income Spending Units	b_i^{53}	29.18*	13.25*	−6.24*	− 9.94*	.78	1.92
	$\sigma_{b_i^{53}}$	(14.41)	(2.22)	(1.58)	(4.93)	(.44)	(1.51)
	b_i^{47}	32.25*	17.08*	−6.52*	−13.98*	1.25*	− .46
	$\sigma_{b_i^{47}}$	(13.19)	(2.94)	(1.41)	(4.57)	(.42)	(2.37)

* Significantly different from zero at a level less than 5 per cent.

TABLE 7—(*cont.*)

		Regression Variables					
		Intercept	Age	Number	Decile	Decile squared	Home Ownership
	b_i^{53-47}	− 3.07	− 3.83	.28	4.04	− .47	2.38
	$\sigma(b_i^{53-47})$	(19.54)	(3.68)	(2.12)	(6.72)	(.61)	(2.81)
	b_i^{53}	19.53	22.63*	—	−15.04	1.35	−11.38*
	$\sigma_{b_i^{53}}$	(51.78)	(8.47)		(18.85)	(1.42)	(4.91)
High Income Single-Person Spending Units	b_i^{47}	154.18	30.39*	—	−73.45*	6.82*	− 6.96
	$\sigma_{b_i^{47}}$	(86.76)	(12.78)		(23.63)	(2.01)	(6.78)
	b_i^{53-47}	−134.65	− 7.76	—	58.41*	−5.47*	− 4.43
	$\sigma(b_i^{53-47})$	(101.04)	(15.33)		(28.45)	(2.46)	(8.37)
	b_i^{53}	63.91*	4.10	—	−21.83*	2.19	—
	$\sigma_{b_i^{53}}$	(22.33)	(9.94)		(10.69)	(1.36)	
Temp. Low Income Single-Person Spending Units	b_i^{47}	2.13	− .94	—	12.48	−1.10	—
	$\sigma_{b_i^{47}}$	(24.95)	(9.82)		(10.97)	(1.18)	
	b_i^{53-47}	61.79	5.04	—	−34.31*	3.29	—
	$\sigma(b_i^{53-47})$	(33.48)	(13.97)		(15.32)	(1.80)	
	b_i^{53}	54.73	13.66*	—	−31.31*	3.83*	1.57
	$\sigma_{b_i^{53}}$	(32.52)	(4.65)		(14.05)	(1.67)	(3.58)
Perm. Low Income Single-Person Spending Units	b_i^{47}	−10.26	23.23*	—	− 7.12	1.33	− .86
	$\sigma_{b_i^{47}}$	(26.67)	(4.83)		(11.66)	(1.32)	(3.93)
	b_i^{53-47}	64.99	− 9.57	—	−24.19	2.50	2.43
	$\sigma(b_i^{53-47})$	(42.06)	(6.70)		(18.26)	(2.13)	(5.32)
	b_i^{53}	37.23	10.41*	−6.76*	− 9.90	.93	.57
	$\sigma_{b_i^{53}}$	(28.71)	(2.65)	(1.75)	(7.56)	(.53)	(1.34)
High Income Multiple-Person Spending Units	b_i^{47}	71.78*	14.58*	−8.11*	−18.15*	1.57*	− 3.02
	$\sigma_{b_i^{47}}$	(29.54)	(4.68)	(2.07)	(7.84)	(.55)	(3.75)
	b_i^{53-47}	−34.55	− 4.17	1.35	8.25	− .64	3.59
	$\sigma(b_i^{53-47})$	(41.19)	(5.38)	(2.71)	(10.89)	(.76)	(3.98)
	b_i^{53}	4.99	.08	−3.06*	4.15*	− .26	− .25
	$\sigma_{b_i^{53}}$	(8.38)	(2.15)	(.86)	(2.57)	(.20)	(1.05)
Temp. Low Income Multiple-Person Spending Units	b_i^{47}	−12.52	8.97	−2.90	8.76	− .41	− 7.01
	$\sigma_{b_i^{47}}$	(29.95)	(7.19)	(2.62)	(8.84)	(.66)	(4.66)
	b_i^{53-47}	17.51	− 8.88	− .16	− 4.61	.14	6.76
	$\sigma(b_i^{53-47})$	(31.10)	(7.51)	(2.76)	(9.21)	(.69)	(4.77)
	b_i^{53}	28.56	11.38*	−6.09*	− 7.05	.48	2.64
	$\sigma_{b_i^{53}}$	(18.68)	(2.44)	(1.85)	(5.60)	(.47)	(1.54)
Perm. Low Income Multiple-Person Spending Units	b_i^{47}	80.44*	10.00*	−7.56*	−23.83*	1.95*	2.27
	$\sigma_{b_i^{47}}$	(18.15)	(4.09)	(1.85)	(5.24)	(.46)	(3.41)
	b_i^{53-47}	−51.88	1.38	1.48	16.78*	−1.48*	.37
	$\sigma(b_i^{53-47})$	(26.04)	(4.77)	(2.62)	(7.67)	(.66)	(3.74)

sults of these tests for specified groups including all groups for which significant F ratios were noted in Table 6.

The decline in the mean LA ratio between 1947 and 1953 for all spending

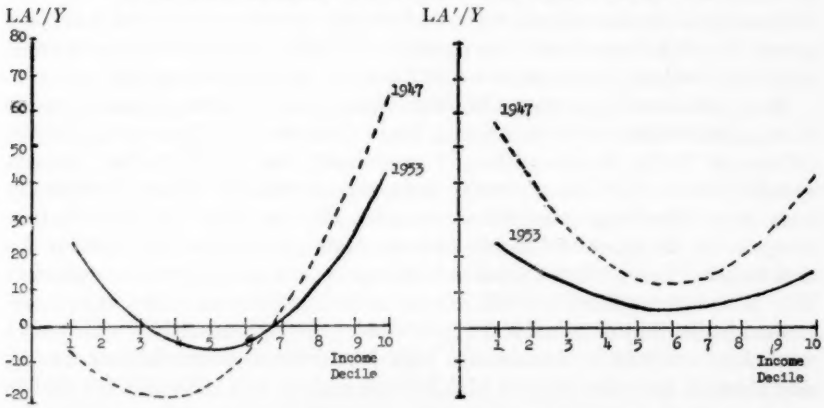
units is partially explained by the significant reduction in the regression coefficient for age. The 1947 LA ratio may therefore be regarded as abnormal for older spending units. Since the high income group consists largely of older spending units, the age effect is consistent with the differences in mean LA ratios between income groups. When single-person spending units are separated from multiple-person spending units, the single-person spending units seem to account for the shift in the LA function of the age variable. The observed change in the regression coefficient for age among multiple-person spending units is negligible; the change for single-person spending units fails by only a small margin to meet the 5 per cent criterion for significant differences between years.

There is no evidence of a shift in the LA ratio function on the income variable for single-person spending units as a group. As shown in Chart 1, the LA ratio functions for 1947 and 1953 intersect, indicating that the 1947 LA ratios may have been abnormal for those with higher incomes but not abnormal for those with lower incomes. The significant shift from the abnormal 1947 levels of the LA ratio for the multiple-person spending units is also shown in Chart 1.

When all spending units are separated into the three income classes, different patterns of variation in the LA ratio between 1947 and 1953 emerge. In the high income group there is a significantly lower slope for age in 1953 than in 1947, again indicating abnormally high LA ratios in 1947 for the older spending units. There is no difference in the age effect between years for either of the two low income groups.

The poor regression fit for 1947 in the temporarily low income group does not allow meaningful interpretation of the significant inter-year difference noted in Table 6 for this group. The inter-year differences in regression coefficients for the permanently low income groups are remarkably small. There

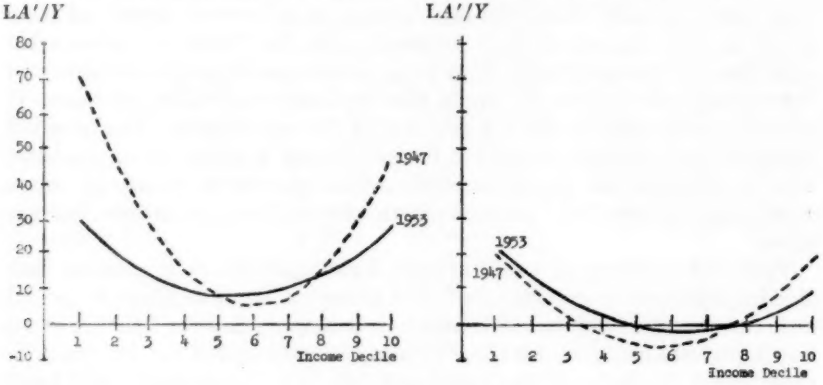
CHART 1. EFFECT OF INCOME ON THE LA RATIO FOR SINGLE- AND MULTIPLE-PERSON SPENDING UNITS, 1947 AND 1953



$$LA'/Y = b_0 + b_3D + b_4D^3$$

Single-Person Spending Units *Multiple-Person Spending Units*

CHART 2. EFFECT OF INCOME ON THE L*A* RATIO FOR HIGH INCOME AND PERMANENTLY LOW INCOME CLASSES, 1947 AND 1953



$$LA'/Y = b_0 + b_3D + b_4D^2$$

High Income Class *Permanently Low Income Class*

is no evidence of any shift in the L*A* function for the permanently low income spending units, and the 1947 L*A* ratios for this group must be regarded as normal in the context of their needs and preferences for liquid reserves.

The effect of the level of income for individual spending units in the income classes is shown in Chart 2 which contrasts the high income class with the permanently low income class. The functions have been plotted for the entire range of income decile because it is possible for a spending unit to have an income which deviates considerably from the mean for his class. It should be noted, however, that the observations tend to cluster above the 7.5 income decile for the high income class and below the 7.5 income decile for the permanently low income class. According to the *t*-tests in Table 7 the inter-year differences in the income effect are statistically significant in the high income group. The right branches of the parabolas for 1947 and 1953 indicate a lower propensity to hold liquid assets in 1953 than in 1947 for this group.

Of the six income-spending unit classes submitted to *F*-tests for net increases in explained variances by separating years, only two tests showed significant differences. In the class consisting of temporarily low income, multiple-person spending units, only one of twelve regression coefficients differs significantly from zero. The large unexplained variance for this class precludes further analysis. In the other of the two classes, single-person spending units in the high income class, there is a significant change from a parabolic income effect in 1947 to no income effect in 1953. Noting again that the most relevant range of income deciles for this group is above the 7.5 level, it appears that World War II conditions resulted in abnormally high liquid asset accumulations among single-person spending units in high income groups. The 1953 data for similar people show that equity in a home is regarded as a favorable substitute for liquid balances and there is no tendency for the L*A* ratio to increase with income.

7. CONCLUSIONS

Perhaps the most important single conclusion to be drawn is that there is no such thing as "the" propensity to hold liquid assets. The evidence indicates that the motivations for holding liquid balances vary widely and that the motivations can be delineated—if not defined—by a demographic and economic classification of consumers.

The L/A ratio is positively related to age in each of the years 1947 and 1953. The rise in the L/A ratio with age has limits, however, as indicated by the abnormal levels attained in 1947 by the older single-person spending units, and by the older high income spending units. Thus the evidence confirms the hypothesis that precautionary reserves increase when spending units need more reserves, are able to cumulate more reserves, and have a lower relative preference for consumption.

A preference for consumption and other non-liquid forms of saving over liquid balances was posited for larger families, and the evidence supports this assumption. The L/A ratio is generally higher among single-person spending units than among larger spending units. Furthermore, the regression coefficient of the size variable is consistently negative. The preference for satisfaction derived from sources other than liquid balances is also suggested by the abnormally high intercept in 1957 for multiple-person spending units.

The effect of the level of income on the propensity to hold liquid assets varies between the groups examined. The L/A ratios of 1947 were abnormally high for spending units who had high incomes. The high income people had evidently passed a point of saturation with liquid reserves, but this conclusion applies particularly to the single-person spending units. In 1953 there was no evidence of a relationship between the L/A ratio and the level of income among the high income, single-person spending units. Moreover, this was the only group showing a tendency to let equity in a home substitute for liquid reserves.

Like the high income spending units, those in the temporarily low income class had abnormally high L/A ratios in 1947. The decline in the L/A ratio from 1947 was greater, in relative terms, for the temporarily low income, multiple-person spending units than for any other group. This sharp decline, along with the very low level of the L/A ratio in 1953, suggests a low propensity to hold liquid assets. As implied by the Modigliani-Brumberg [6] and Friedman [1] theories of saving,[6] the expectation of higher incomes may reduce the need for current saving.

Those in the low income class may have been able to fulfill a latent need for more security in the form of liquid balances during the prosperous periods of World War II and the following years. Their L/A ratios in 1947 could not be considered abnormal relative to comparable L/A ratios in 1953.[7] This finding is reinforced by the data for 1953 for low income, single-person spending units. As income increases among these spending units the L/A ratio declines but then rises in the higher range of income decile. This behavior suggests a general need

[6] See also Harold W. Watts, "Long-run income expectations and consumer saving," *Studies in Household Economic Behavior*, Yale Studies in Economics, Vol. 9, Yale University Press, 1958.

[7] It is possible that the low income class did not accumulate unusually high liquid balances during World War II. In the absence of data to support this belief I choose to interpret the findings as if the permanently low income class had been forced to save.

for the security provided by liquid balances among low income spending units, a need which is partially fulfilled only when the income earner has no dependents and when his income rises beyond the fourth income decile.

The results of this paper differ, in part, from those found in two other recent investigations of liquid asset holdings. Tobin and Watts [8], in the course of an extensive analysis of many stocks and flows measured by the 1950 Budget Study, Bureau of Labor Statistics, calculated regressions for "cash in bank" and changes therein. They found that variations in both stocks and flows of this component of liquid assets were associated with differences in age, size of family, disposable income, and housing level as well as some other variables not included in this analysis. After removing demographic effects they found that cash balances and flows are related to stock of durable goods, mortgage debt, and installment debt.

H. F. Lydall [5] has emphasized net worth as a determinant of liquid asset holdings. He concludes from an analysis of British spending units that net worth is much stronger than income in determining the level of liquid asset holdings. The effects of demographic variables were not considered explicitly.

The propensity to hold liquid assets is by no means simply determined. At least three major kinds of variables are involved: demographic characteristics of the household, the stock of non-liquid assets, and the current and expected income position of the household. The three types of variables are themselves highly inter-correlated and there may be complex interaction effects among them. We still lack a theory which will explain satisfactorily the level of liquid asset holdings and the relationship between these reserves and saving behavior.

APPENDIX

Income deciles are observed for individual spending units in each Survey of Consumer Finances. The income deciles are established by arraying absolute incomes and dividing the array into tenths. The income deciles range from 1 for the lowest 10 per cent of incomes to 10 for the highest 10 per cent of incomes. Tables 1*A* through 1*C* show the weighted means of income deciles for groups of spending units classified by race, region, occupation, education, and age.

Data for four years, viz. 1947, 1948, 1952, and 1953, have been combined in constructing these tables. Tests performed by Robert Summers indicate that differences between group means for individual years are not statistically significant.

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TABLE 1A. MEAN VALUES OF INCOME DECILE POSITION FOR SELECTED SPENDING UNITS 1947–48 AND 1952–53[1]

Education	White, South, Non-Farm				
	Professional, Managerial, Business, Clerical	Skilled	Unskilled	Other	Retired
Grade-School or less: Age					
18–24		4.5–.3 .6/74			
25–34		5.3–.2 .8/95			
35–44		5.1–.2 1.0/109			
45–54		5.3–.2 .7/71			
55–64		5.4–.3 .5/66			2.3–.2 1.1/30
65+					
High-School: Age					
18–24	3.5–.2 .6/72	4.7–.3 .5/61	3.1–.3 .2/23	4.1–.5 .2/20	
25–34	6.6–.2 1.2/145	6.4–.2 .9/112	3.7–.3 .5/56	4.7–.4 .4/45	
35–44	6.6–.2 1.2/162	6.5–.2 .7/87	3.7–.2 .6/62	5.1–.5 .4/38	
45–54	6.9–.2 .9/137	7.0–.4 .4/49	3.6–.3 .4/42	3.5–.4 .4/49	
55–64	6.3–.2 1.1/148	5.8–.5	2.4–.3 .2/20	2.8–.3 .4/49	
65+	5.2–.5 .3/42	.3/32	1.7–.2 .2/19	2.3–.2 .6/72	3.2–.4 .3/36

TABLE 1*A*—(*cont.*)

Education	White, South, Non-Farm				
	Professional, Managerial, Business, Clerical	Skilled	Unskilled	Other	Retired
College: Age					
18–24	3.8–.3 .3/38				
25–34	7.4–.2 .9/128				
35–44	7.9–.2 .8/127				
45–54	8.4–.2 .4/79				
55–64	7.2–.4				
65+	.4/66				3.7–.5 .2/22

[1] Entries in Table 1 consist of $A-B$, C/D, where A = mean value of income decile; B = standard error of mean income decile; C = proportion of sample represented in the cell; D = number of cases.

TABLE 1*B*. MEAN VALUES OF INCOME DECILE POSITION FOR SELECTED SPENDING UNITS, 1947–48 AND 1952–53[1]

Education	White, Non-South, Non-Farm							
	Professional	Business Owners	Managers, Officials	Clerical and Sales	Skilled	Unskilled	Other	Retired
Grade-School or less: Age								
18–24					4.2–.4 .3/28	3.8–.5 .2/24		
25–34					6.3–.2 1.4/159	4.9–.3 .4/47		
35–44					6.8–.1 2.0/258	5.1–.2 .7/80		
45–54					6.9–.1 2.3/286	4.9–.2 .9/112		
55–64					6.3–.1 1.9/242	4.0–.2 1.0/124		
65+					5.7–.3 .5/59	3.3–.2 .6/70		2.7–.1 2.7/326

TABLE 1*B*—(*cont.*)

Education	White, Non-South, Non-Farm							
	Profes-sional	Business Owners	Managers, Officials	Clerical and Sales	Skilled	Un-skilled	Other	Retired
High-School: Age								
18–24	3.3–.6 .4/7	6.3–.9 .1/9	3.9–.5 .1/8	3.6–.1 1.4/202	5.0–.2 1.2/146	4.1–.3 .5/63	3.8–.3 .3/32	
25–34	6.9–.3 .3/39	7.1–.2 .7/98	7.7–.3 .3/43	6.3–.1 1.8/239	6.8–.1 3.7/451	5.3–.2 .7/80	5.1–.3 .5/63	
35–44	7.4–.3 .4/50	7.7–.2 1.1/165	7.8–.3 .4/58	6.7–.1 1.4/209	7.3–.1 3.0/382	5.6–.3 .5/56	5.2–.3 .5/68	
45–54	7.2–.5 .2/37	7.5–.2 .9/143	7.4–.3 .5/75	6.9–.2 1.3/183	7.4–.1 1.5/200	4.9–.3 .4/47	4.2–.3 .8/98	
55–64	7.1–.5	6.5–.3 .7/106	6.9–.4 .3/46	6.3–.2 .8/120	7.4–.2 .7/93	4.7–.3 .3/40	3.2–.2 1.3/161	
65+	.2/27	5.3–.4 .3/47	5.3–.7 .1/20	5.4–.5 .2/27	5.5–.5 .1/13	3.1–.4 .2/21	2.2–.1 1.6/197	3.4–.2 .7/95
College: Age								
18–24	5.4–.3 .4/53	**	7.8–1.1 */5	4.2–.3 .3/52	5.6–.6 .2/20		3.4–.3 .4/61	
25–34	7.8–.1 1.3/188	9.0–.1 .3/51	8.3–.2 .4/63	6.6–.2 .8/120	7.8–.2 .4/57		5.3–.5 .2/24	
35–44	8.9–.1 1.1/203	8.9–.2 .4/73	9.0–.2 .4/73	7.6–.2 .5/79	7.6–.3 .3/47		7.1–.7 .1/16	
45–54	8.2–.2 .9/166	9.1–.3 .3/58	9.2–.2 .4/64	6.9–.3 .6/94	7.6–.5 .2/27		6.5–.8 .1/20	
55–64	8.3–.3 .4/74	8.9–.3	9.0–.3	7.4–.3	7.2–.3 .1/17			
65+	7.3–.4 .2/39	.3/48	.3/50	.3/41	**		4.5–.6 .2/33	5.1–.4 .4/74

* Less than one-half of one per cent.

** No observations.

[1] Entries in Table 1 consist of A-B, C/D, where A = mean value of income decile; B = standard error of mean income decile; C = proportion of sample represented in the cell; D = number of cases.

TABLE 1C. MEAN VALUES OF INCOME DECILE POSITION FOR SELECTED SPENDING UNITS 1947–8 AND 1952–3[1]

Age	Not White				Farmers	
	South		Not South			
	Grade School or less	High School or more	Grade School or less	High School or more	Grade School or less	High School or more
18–24	2.2–.2 .3/26	2.9–.4 .3/27	4.1–.9 .1/9	3.5–.3 .4/40	2.7–.5 .2/26	4.0–.5 .2/28
25–34	3.3–.2 .5/48	4.0–.3 .5/51	4.4–.3 .4/35	5.3–.2 .7/76	3.7–.3 .7/109	4.4–.3 .6/94
35–44	3.3–.2 .7/70	4.5–.6 .2/21	4.4–.3 .6/57	5.6–.4 .4/51	4.0–.2 1.3/205	4.8–.3 .8/113
45–54	3.0–.2 .6/58		3.9–.3 .6/58		3.8–.2 1.4/204	5.7–.4 .5/74
55–64	2.5–.3 .4/40	3.6–.6 .1/16	4.2–.5 .3/28	5.4–.4 .4/42	3.5–.2 1.3/213	5.2–.5 .3/49
65+	1.6–.2 .6/55		1.9–.3 .2/24		2.9–.2 .9/149	3.9–.6 .2/29

[1] Entries in Table 1 consist of $A-B, C/D$, where A = mean value of income decile; B = standard error of mean income decile; C = proportion of sample represented in the cell; D = number of cases.

EARLY FAILURES IN LIFE TESTING

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Data from certain life test experiments exhibit such an unusually high concentration of failures near time 0 that the assumption of an over-all exponential density is unwarranted. One hypothesis for this phenomenon is that due to faulty construction or defective parts, certain test items fail prematurely; these items are termed "early failures." To handle this type of data, an early failure model is postulated in which one failure rate is assumed to be in effect for an initial time interval $[0, T_0)$ and another, lower failure rate is operative thereafter. Estimators for the two failure rates are given in the case where T_0 is known and in the case where T_0 is not known exactly but can be assumed to be within a specified interval. Methods for obtaining approximate large sample confidence regions are outlined and procedures for handling small samples are described.

1. INTRODUCTION

IN THEIR fundamental paper, B. Epstein and M. Sobel [4] postulated a model for life test experiments and exhibited means by which such experiments could be analyzed statistically. To summarize briefly, a life test experiment consists in testing N components or items (whether they be radio tubes, capacitors, etc.) until n of them have failed, and the basic assumption is that the probability density for length of life is exponential with parameter $1/\theta$, i.e.,

$$p_T(t;\theta)dt = \frac{1}{\theta} e^{-t/\theta}dt, \qquad t \geqq 0. \tag{1}$$

The maximum likelihood estimator of θ is

$$\hat{\theta}_{nN} = \frac{t_{1N} + t_{2N} + \cdots + t_{nN} + (N-n)t_{nN}}{n}, \tag{2}$$

where $t_{1N} \leqq \cdots \leqq t_{nN}$ are the ordered times of failure. The desirable properties of $\hat{\theta}_{nN}$ are numerous. In particular, $\hat{\theta}_{nN}$ is exactly distributed as $(\theta/2n)\chi^2_{2n}$, and it is a sufficient, efficient minimum variance unbiased estimator of θ. These results and the corresponding hypothesis testing techniques based on $\hat{\theta}_{nN}$ are discussed in reference [4].

This model has been extensively applied, and the consensus is that the exponential model is a reasonable assumption in many instances. However, frequently the experimental data indicate that the exponential model fails to hold exactly because the experimenter is confronted with "early failures." The data suggest the hypothesis that the items under test are composed of two groups. There are those which have been constructed according to specification and will eventually fail randomly; the life span probability density for these items is exponential. On the other hand, there are also those components which, due to inferior quality or faulty construction, will fail shortly after being put on

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test. These "early failures" preclude the assumption of an exponential density and the use of existing techniques to estimate the mean life of "standard" components.

This paper represents an attempt to formulate a probability model in which it is possible statistically to handle early failures. However, it by no means constitutes a solution to all early failure problems. As with all probability models, whether or not it can be applied to a particular experimental problem should be decided by scrutiny of the assumptions underlying the experimental process and the agreement between the theoretical and sample distributions.

2. EARLY FAILURE MODEL

The exponential density (1) assumed by Epstein and Sobel arises from considerations on the rate of failure of test items. For a component under test the failure rate at time t, $\lambda(t)$, is defined by

$$\lim_{\Delta t \to 0} \frac{P\{T > t\} - P\{T > t + \Delta t\}}{\Delta t \cdot P\{T > t\}} = \lambda t, \tag{3}$$

where the random variable T is the time of failure. The limit expression (3) indicates that the conditional probability of the item failing in an infinitesimal interval of length Δt after t, given it has survived to time t, is approximately $\lambda(t)\Delta t$, or, equivalently,

$$P\{T > t + \Delta t\} = (1 - \lambda(t)\Delta t)P\{T > t\} + o(\Delta t). \tag{4}$$

$\lambda(t)$ is the parameter of a possibly nonhomogeneous pure death process (see [5], Ch. XVII).

If it is assumed that $\lambda(t) = \lambda = 1/\theta$ for all t, a differential equation argument shows the probability density for time to failure is given by (1). Thus, the assumption of an exponential density is equivalent to the assumption of a constant failure rate.

The early failure model permits the failure rate to vary with t in the simplest possible fashion, namely,

$$\lambda(t) = \begin{cases} \lambda_1, & 0 \leqq t < T_0, \\ \lambda_2, & T_0 \leqq t, \end{cases} \tag{5}$$

where $\lambda_1 > \lambda_2$. An analogous differential equation argument shows that the probability density corresponding to this failure rate is

$$p_T(t; \lambda_1, \lambda_2)dt = \begin{cases} \lambda_1 e^{-\lambda_1 t}dt, & 0 \leqq t < T_0, \\ \lambda_2 e^{-\lambda_1 T_0 - \lambda_2(t - T_0)}dt, & T_0 \leqq t. \end{cases} \tag{6}$$

The analysis throughout the remainder of the paper will be in terms of λ_1 and λ_2, although it could just as well be performed in terms of $\theta_1 = 1/\lambda_1$ and $\theta_2 = 1/\lambda_2$. The choice of λ_1 and λ_2 was prompted by little more than notational simplicity.

The intuitive justification for the early failure model follows from the fact that when a component is put on test, it is not known whether it is an "early failure" or a "standard" item. Since some will be early failures, the failure rate

on the average will be high at the start, but if an item has survived for a certain period of time (T_0), then it cannot be an "early failure" so its failure rate will be lower for the succeeding time period.

An alternative model is to assume that the population of components is composed of two groups with different failure rates, λ_1 and λ_2, and the early failure group comprises a proportion p of this population. In this model the estimation of λ_1, λ_2, and p is complicated unless each failure is examined to determine to which sub-population it belongs; for this latter approach, the reader is referred to a recent article by W. Mendenhall and R. J. Hader [6]. For some experimental programs, however, it is not always possible to examine each failure to determine its cause.

The particular test structure in which the early failure model will be applied is the following: N items are put on test and are tested until failure or time T_1, whichever is sooner, where $T_1 > T_0$. Due to limitations on experimental time, no item is tested past T_1. For this censored test structure, let T^* be the time at which the test terminates for any particular unit. For $T^* = t < T_1$, the value $T^* = t$ denotes the failure of the item at time t. $T^* = T_1$ denotes the item had not failed when the experiment was terminated at T_1. The early failure model determines the probability distribution of T^* to be

$$\begin{aligned} p_{T^*}(t; \lambda_1, \lambda_2)dt &= \lambda_1 e^{-\lambda_1 t}dt, && 0 \leqq t < T_0, \\ &= \lambda_2 e^{-\lambda_1 T_0 - \lambda_2 (t - T_0)}dt, && T_0 \leqq t < T_1, \\ P\{T^* = T_1\} &= e^{-\lambda_1 T_0 - \lambda_2 (T_1 - T_0)}. \end{aligned} \tag{7}$$

In the event that the experimental procedure is to test all items to failure, the above expressions and all succeeding expressions become the limiting expressions as $T_1 \to \infty$.

3. LARGE SAMPLE THEORY

If N components or items are tested to failure or time T_1, the experimental data consist of an ordered sample $(t_1, \cdots, t_n, T_1, \cdots, T_1)$ where $t_1 \leqq \cdots \leqq t_n$ are the ordered failure times (prior to T_1) and the $N-n$ T_1's represent the $N-n$ components surviving at T_1. If the experimenter assumes the probability density of time to failure is exponential with parameter $\lambda = 1/\theta$, the maximum likelihood estimator of λ is

$$\hat{\lambda} = \frac{n}{\sum_1^n t_i + (N - n)T_1}. \tag{8}$$

As $N \to \infty$, $\hat{\lambda}$ will converge in probability to λ; i.e., for all $\epsilon > 0$, $P\{|\hat{\lambda} - \lambda| > \epsilon\} \to 0$ as $N \to \infty$. However, if the assumption of a constant failure rate is incorrect and in fact the early failure model pertains, $\hat{\lambda}$ will converge in probability to

$$\Lambda(\lambda_1, \lambda_2, T_0, T_1) = \frac{1 - e^{-\lambda_1 T_0 - \lambda_2 (T_1 - T_0)}}{\frac{1}{\lambda_1}(1 - e^{-\lambda_1 T_0}) + \frac{1}{\lambda_2} e^{-\lambda_1 T_0}(1 - e^{-\lambda_2 (T_1 - T_0)})}, \tag{9}$$

(see Appendix). By choosing plausible values of λ_1, λ_2, and T_0, the experimenter can determine from (9) the magnitude of the error he may be making by not assuming the early failure model.

In the estimation problem for the early failure model it is necessary to distinguish two principal cases.

Case I. T_0 is known.

Let $(t_1, \cdots, t_r, t_{r+1}, \cdots, t_n, T_1, \cdots, T_1)$ be an ordered sample of N observations whose common distribution is the distribution (7). The observations are ordered in the manner $t_1 \leqq \cdots \leqq t_r < T_0 \leqq t_{r+1} \leqq \cdots \leqq t_n < T_1$, and the final sequence of T_1's represents the $N-n$ items which survived the test. For this sample, the maximum likelihood estimates of λ_1 and λ_2 are

$$\hat{\lambda}_1 = \frac{r}{\sum_1^r t_i + (N-r)T_0}, \tag{10}$$

$$\hat{\lambda}_2 = \begin{cases} \dfrac{n-r}{\sum_{r+1}^{n} (t_i - T_0) + (N-n)(T_1 - T_0)}, & r < N, \\ \text{undefined}, & r = N. \end{cases} \tag{11}$$

The case of primary interest is $0 < r < N$ since the probability of the complementary case tends to zero as $N \to +\infty$.

(10) and (11) give point estimates of λ_1 and λ_2. The exact distribution of $(\hat{\lambda}_1, \hat{\lambda}_2)$ does not have a simple form, and consequently it is not feasible to determine an exact confidence region for (λ_1, λ_2). However, in the Appendix it is shown that as $N \to \infty$, the estimators $\hat{\lambda}_1$ and $\hat{\lambda}_2$ have the following properties:

(*i*) $\hat{\lambda}_1$ is normally distributed with mean λ_1 and variance

$$\frac{\lambda_1^2}{N(1 - e^{-\lambda_1 T_0})}, \tag{12*i*}$$

(*ii*) $\hat{\lambda}_2$ is normally distributed with mean λ_2 and variance

$$\frac{\lambda_2^2}{Ne^{-\lambda_1 T_0}(1 - e^{-\lambda_2 (T_1 - T_0)})}, \tag{12*ii*}$$

(*iii*) $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are independently distributed. (12*iii*)

By virtue of (*i*)–(*iii*) for N sufficiently large, an approximate confidence region with confidence coefficient $1-\alpha$ consists of those points (λ_1, λ_2) satisfying the following two sets of inequalities:

$$\begin{aligned} -K_\alpha &\leqq \frac{\sqrt{N}(\hat{\lambda}_1 - \lambda_1)(1 - e^{-\lambda_1 T_0})^{1/2}}{\lambda_1} \leqq K_\alpha, \\ -K_\alpha &\leqq \frac{\sqrt{N}(\hat{\lambda}_2 - \lambda_2)e^{-\lambda_1 T_0/2}\left[1 - e^{-\lambda_2 (T_1 - T_0)}\right]^{1/2}}{\lambda_2} \leqq K_\alpha, \end{aligned} \tag{13}$$

where K_α is defined by

$$\sqrt{1-\alpha} = \frac{1}{\sqrt{2\pi}} \int_{-K_\alpha}^{K_\alpha} e^{-x^2/2} dx. \tag{14}$$

One-sided confidence regions can be constructed similarly. In each case, it can be easily shown that the confidence region is simply connected in the (λ_1, λ_2)-plane.

A confidence ellipsoid constructed from the χ^2_2 distribution also yields a confidence region, but it is harder to determine numerically than the above region.

Case II. T_0 is not known exactly, but $T_2 \leqq T_0 \leqq T_3$ for known values T_2 and T_3.

The jump discontinuity in the failure rate $\lambda(t)$ at T_0 is somewhat artificial and it may therefore be very difficult for the experimenter to designate an exact value for T_0. However, he may be able and willing to make the following statement: If a component or an item fails before T_2, it is most likely a dud or early failure, but if it survives to time T_3, the item cannot reasonably be classified as such. Accumulated evidence will very likely indicate values for T_2 and T_3. For the estimators of λ_1 and λ_2 which are proposed in this section, the exact value of T_0 is immaterial so long as it lies in the interval $[T_2, T_3]$.

For an ordered sample of N observations

$$(t_1, \cdots, t_r, t_{r+1}, \cdots, t_s, t_{s+1}, \cdots, t_n, T_1, \cdots, T_1)$$

where

$$t_1 \leqq \cdots \leqq t_r < T_2 \leqq t_{r+1} \leqq \cdots \leqq t_s \leqq T_3 < t_{s+1} \leqq \cdots \leqq t_n < T_1,$$

the maximum likelihood estimators (10) and (11) of λ_1 and λ_2 are no longer actual estimators since T_0 is not known exactly. One possibility is to estimate simultaneously T_0 by the principle of maximum likelihood or, perhaps, quasi-maximum likelihood (see [1]). However, the computations involved in estimating T_0 are extremely laborious. This, coupled with the possible artificiality of T_0, lends cogency to the following alternative scheme. Disregard the observations $t_{r+1}, \cdots, t_s$ in the interval $[T_2, T_3]$ and estimate λ_1 and λ_2 by

$$\hat{\lambda}_1 = \frac{r}{\sum_1^r t_i + (N-r)T_2}, \tag{15}$$

$$\hat{\lambda}_2 = \begin{cases} \dfrac{n-s}{\sum_{s+1}^{n} (t_i - T_3) + (N-n)(T_1 - T_3)}, & s < N, \\ \text{undefined} & s = N. \end{cases} \tag{16}$$

The case of primary interest is $r > 0$ and $s < N$ since the probability of the complementary case tends to zero as $N \to \infty$.

$\hat{\lambda}_1$ and $\hat{\lambda}_2$ are not maximum likelihood estimators, but they are consistent and asymptotically jointly normally distributed. As in the previous case, the exact

distribution of $(\hat{\lambda}_1, \hat{\lambda}_2)$ for finite sample size N is not simple, but for large samples the asymptotic normality can be employed to obtain an approximate confidence region for (λ_1, λ_2). Precisely, as $N \to \infty$,

(*i*) $\hat{\lambda}_1$ is normally distributed with mean λ_1 and variance

$$\frac{\lambda_1^2}{N(1 - e^{-\lambda_1 T_2})}, \tag{17i}$$

(*ii*) $\hat{\lambda}_2$ is normally distributed with mean λ_2 and variance

$$\sigma^2 \leqq \frac{\lambda_2^2}{Ne^{-\lambda_1 T_2}(1 - e^{-\lambda_2(T_1 - T_2)})}, \tag{17ii}$$

(*iii*) $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are independently distributed. (17*iii*)

For N sufficiently large, an approximate confidence region with confidence coefficient $\geqq 1 - \alpha$ consists of those points (λ_1, λ_2) satisfying the following two sets of inequalities:

$$\begin{aligned} -K_\alpha &\leqq \frac{\sqrt{N}(\hat{\lambda}_1 - \lambda_1)(1 - e^{-\lambda_1 T_2})^{1/2}}{\lambda_1} \leqq K_\alpha, \\ -K_\alpha &\leqq \frac{\sqrt{N}(\hat{\lambda}_2 - \lambda_2)e^{-\lambda_1 T_2/2}(1 - e^{-\lambda_2(T_1 - T_2)})^{1/2}}{\lambda_2} \leqq K_\alpha, \end{aligned} \tag{18}$$

where K_α is defined by (14).

In this section on large sample theory it has been suggested that for N sufficiently large, approximate confidence regions may be constructed with the aid of the normal distribution. The question which naturally arises is, how large is "sufficiently large?" No results have been obtained on how fast the approximations of this section approach the true situation. There is an article, however, by W. L. Deemer, Jr., and D. F. Votaw, Jr. [3] in which a sampling experiment was performed on a censored exponential distribution. The outcome indicates that for sample sizes of 100 the analogous approximations are fairly good. This, of course, says nothing about the approximations under consideration here, but it would be rather surprising if the rates of convergence were quite dissimilar.

4. SMALL SAMPLE THEORY

The problem of what to do when the sample is small is invariably messier and more difficult than the corresponding large sample problem. Also, any solutions or partial solutions are less satisfying, both mathematically and aesthetically. Nevertheless, the problem still remains.

One of the more sensible solutions for the early failure problem is to replace the probability density by a multinomial distribution whose probabilities are determined by (6) and the intervals delineated by the points T_0, T_1, T_2, and T_3.

Case I. T_0 is known.

Let

$$p_1 = P\{0 \leqq T < T_0\} = 1 - e^{-\lambda_1 T_0},$$
$$p_2 = P\{T_0 \leqq T < T_1\} = e^{-\lambda_1 T_0}[1 - e^{-\lambda_2 (T_1 - T_0)}], \quad \text{and} \tag{19}$$
$$p_3 = P\{T \geqq T_1\} = e^{-\lambda_1 T_0 - \lambda_2 (T_1 - T_0)}.$$

For an ordered sample $(t_1, \cdots, t_r, t_{r+1}, \cdots, t_n, T_1, \cdots, T_1)$ of N independent observations, the estimates of p_1, p_2, and p_3 are

$$\hat{p}_1 = \frac{r}{N}, \qquad \hat{p}_2 = \frac{n - r}{N}, \quad \text{and} \quad \hat{p}_3 = \frac{N - n}{N}.$$

If these estimates are substituted into (19) and any two of the resulting equations are solved for λ_1 and λ_2, the estimates

$$\hat{\lambda}_1 = \frac{-\log (1 - \hat{p}_1)}{T_0}, \qquad \hat{\lambda}_2 = \frac{-\log \left(\dfrac{\hat{p}_3}{1 - \hat{p}_1} \right)}{T_1 - T_0}, \tag{20}$$

of λ_1 and λ_2 are obtained. (Note that the estimates of Section 3 are the maximum likelihood estimates for every N, but for small N no confidence region using them can be constructed conveniently.)

To construct a confidence region for (λ_1, λ_2), it is sufficient to put a confidence region on any two of p_1, p_2, or p_3 by using $\hat{p}_1$, $\hat{p}_2$, or $\hat{p}_3$ and the trinomial distribution, and then solve for those values of (λ_1, λ_2) which give values of p_1, p_2 or p_3 in this region. Unfortunately, the trinomial distribution is not tabulated. Two alternatives can be suggested. A confidence interval can be put on any one of p_1, p_2, or p_3 by using the corresponding $\hat{p}_1$, $\hat{p}_2$, or $\hat{p}_3$ and the binomial distribution. Or two regions on two different single parameters p_1, p_2, or p_3 can be combined according to the rule $P\{A \cap B\} \geqq 1 - P\{A^c\} - P\{B^c\}$. In this latter case, the confidence coefficient is not known exactly but is bounded from below.

If an exact confidence coefficient $1 - \alpha$ is desired, a reasonable procedure is to place the confidence interval on p_3. The resulting confidence region for (λ_1, λ_2) consists of those values which satisfy

$$-\log \bar{p}_\alpha(\hat{p}_3) \leqq \lambda_1 T_0 + \lambda_2 (T_1 - T_0) \leqq -\log \underline{p}_\alpha(\hat{p}_3), \tag{21}$$

where $\bar{p}_\alpha(\hat{p}_3)$ and $\underline{p}_\alpha(\hat{p}_3)$ are the upper and lower $100(1-\alpha)\%$ confidence limits, respectively, for the binomial parameter $p(=p_3)$ when the estimate is $\hat{p}(=\hat{p}_3)$.

If it suffices to have a confidence region with confidence coefficient $\geqq 1 - \alpha$, then it is preferable to combine the regions on p_1 and p_3. This leads to a confidence region consisting of those values of (λ_1, λ_2) which simultaneously satisfy

$$\frac{-\log (1 - \underline{p}_\nu(\hat{p}_1))}{T_0} \leqq \lambda_1 \leqq \frac{-\log (1 - \bar{p}_\nu(\hat{p}_1))}{T_0},$$
$$-\log \bar{p}_\nu(\hat{p}_3) \leqq \lambda_1 T_0 + \lambda_2 (T_1 - T_0) \leqq -\log \underline{p}_\nu(\hat{p}_3), \tag{22}$$

where $\nu = \alpha/2$.

Case II. T_0 is not known exactly, but $T_2 \leqq T_0 \leqq T_3$.

Let

$$\begin{aligned} p_1 &= P\{0 \leqq T < T_2\} = 1 - e^{-\lambda_1 T_2}, \\ p_2 &= P\{T_2 \leqq T \leqq T_3\} = e^{-\lambda_1 T_2}(1 - p), \\ p_3 &= P\{T_3 < T < T_1\} = pe^{-\lambda_1 T_2}[1 - e^{-\lambda_2(T_1 - T_3)}], \quad \text{and} \\ p_4 &= P\{T \geqq T_1\} = pe^{-\lambda_1 T_2 - \lambda_2(T_1 - T_3)}, \end{aligned} \tag{23}$$

where p is actually equal to $P\{T > T_3 | T \geqq T_2\} = e^{-\lambda_1(T_0 - T_2) - \lambda_2(T_3 - T_0)}$. For an ordered sample $(t_1, \cdots, t_r, t_{r+1}, \cdots, t_s, t_{s+1}, \cdots, t_n, T_1, \cdots, T_1)$, the estimates of λ_1 and λ_2 derived from the binomial estimates are

$$\hat{\lambda}_1 = \frac{-\log(1 - \hat{p}_1)}{T_1}, \qquad \hat{\lambda}_2 = \frac{\log\left(1 + \dfrac{\hat{p}_3}{\hat{p}_4}\right)}{T_1 - T_3}. \tag{24}$$

The problem of choosing a confidence region for (λ_1, λ_2) is more difficult than that encountered for T_0 known since here the distribution is quadrinomial. The simplest and perhaps most reasonable region with confidence coefficient $\geqq 1 - \alpha$ is the set of all (λ_1, λ_2) satisfying

$$-\log \bar{p}(\hat{p}_4) \leqq \lambda_1 T' + \lambda_2(T_1 - T') \leqq -\log \underline{p}(\hat{p}_4) \tag{25}$$

for some T' in $[T_2, T_3]$. This confidence region is generated by placing a binomial confidence interval on the parameter p_4, letting p vary over its natural range $[\exp(-\lambda_1(T_3 - T_2)), \exp(-\lambda_2(T_3 - T_2))]$, and projecting the resulting region onto the (λ_1, λ_2)-plane. It has probability $\geqq 1 - \alpha$ of containing the true point (λ_1, λ_2) regardless of the value of T_0.

5. RELIABILITY

The reliability of a component or item is usually defined to be the probability that it will function for a specified length of time, or equivalently, the probability it will not fail in this period. Precisely, let $R(t)$ be the probability that an item will survive for t units of time.

In the case where $t < T_0$ and nothing has been done to eliminate early failures, a reasonable estimate of $R(t)$ is $\exp(-\hat{\lambda}_1 t)$ where $\hat{\lambda}_1$ is either given by (10), (15), (20), or (24). A confidence interval for $R(t)$ with confidence coefficient $1 - \alpha$ is given by $(\exp(-\bar{\lambda}_1 t), \exp(-\underline{\lambda}_1 t))$ where $\bar{\lambda}_1$ and $\underline{\lambda}_1$ are the upper and lower confidence limits, respectively, on λ_1 determined from either (12i), (17i), or p_1.

Because of the reproductive property of the exponential density, a possible procedure of eliminating early failures is to pre-test or burn in all components past the change point T_0. If the failure rate is constant above T_0, all pre-tested items will have an exponential density with parameter λ_2 for time to failure. A reasonable estimate of $R(t)$ now is $\exp(-\hat{\lambda}_2 t)$, and a confidence interval with confidence coefficient $\geqq 1 - \alpha$ can be constructed by the following procedure. Project the confidence region for (λ_1, λ_2) generated by (13), (18), (21), (22), or (25) onto the λ_2-axis. This will produce two limits, $\bar{\lambda}_2$ and $\underline{\lambda}_2$, for λ_2 which determine a confidence interval $(\exp(-\bar{\lambda}_2 t), \exp(-\underline{\lambda}_2 t))$ for $R(t)$.

The case in which $t > T_0$ and the components have not been pre-tested is more complex since $R(t)$ involves λ_1, λ_2, and T_0. If T_0 is known, $\exp(-\hat{\lambda}_1 T_0 - \hat{\lambda}_2(t - T_0))$ estimates $R(t)$ and a confidence interval with confidence coefficient $\geqq 1-\alpha$ is constructed by determining the range of values of $\lambda_1 T_0 + \lambda_2(t - T_0)$ as (λ_1, λ_2) varies over its confidence region. If T_0 is unknown, no estimate of $R(t)$ depending on the observations only through $\hat{\lambda}_1$ and $\hat{\lambda}_2$ exists, but $(\exp(-\bar{\lambda}(T_3)), \exp(-\underline{\lambda}(T_2))$ does constitute a confidence interval for $R(t)$ with confidence $\geqq 1-\alpha$, where

$$\underline{\lambda}(T_2) = \min_{(\lambda_1,\lambda_2)} [\lambda_1 T_2 + \lambda_2(t - T_2)], \qquad \bar{\lambda}(T_3) = \max_{(\lambda_1,\lambda_2)} [\lambda_1 T_3 + \lambda_2(t - T_3)],$$

and the min(max) are taken over the confidence region on (λ_1, λ_2).

6. APPENDIX

1. $\hat{\lambda} \xrightarrow{p} \Lambda$.

By definition, $\hat{\lambda} = n/(\sum_1^n t_i + (N-n)T_1)$ where $(t_1, \cdots, t_n, T_1, \cdots, T_1)$ is an ordered sample of N independent observations on the distribution (6) censored at T_1. This can be rewritten as

$$\hat{\lambda} = \frac{n}{\sum_1^N t_i^*} = \frac{n}{N} \cdot \frac{N}{\sum_1^N t_i^*},$$

where $(t_1^*, \cdots, t_N^*)$ is an unordered sample of N independent observations on the distribution (7). The random variable n has a binomial distribution with parameters N and

$$p = 1 - e^{-\lambda_1 T_0 - \lambda_2(T_1 - T_0)},$$

and $t_1^*, \cdots, t_N^*$ are independent observations on a distribution with mean

$$E(T^*) = \frac{1}{\lambda_1}(1 - e^{-\lambda_1 T_0}) + \frac{1}{\lambda_2} e^{-\lambda_1 T_0}(1 - e^{-\lambda_2(T_1 - T_0)}).$$

Therefore, as $N \to \infty$, $n/N \xrightarrow{p} p$ and $\sum_1^N t_i^*/N \xrightarrow{p} E(T^*)$, so by Slutsky's theorem (see [2]), $\hat{\lambda} \xrightarrow{p} p/E(T^*) = \Lambda(\lambda_1, \lambda_2; T_0, T_1)$.

Note the convergence is also almost sure.

2. Asymptotic normality of $(\hat{\lambda}_1, \hat{\lambda}_2)$; T_0 known.

The likelihood equation for the ordered sample $(t_1, \cdots, t_r, t_{r+1}, \cdots, t_n, T_1, \cdots, T_1)$ of N independent observations on the distribution (6) censored at T_1 is

$$L = \frac{N!}{(N-n)!} \lambda_1^r \lambda_2^{n-r} \exp\left[-\lambda_1 \sum_1^r t_i\right] \cdot \exp\left[-(n-r)\lambda_1 T_0 - \lambda_2 \sum_{r+1}^n (t_i - T_0)\right] \cdot \exp\left[-(N-n)\lambda_1 T_0 - (N-n)\lambda_2(T_1 - T_0)\right].$$

It is readily verified that the estimates $\hat{\lambda}_1$, $\hat{\lambda}_2$ given by (10), (11) maximize L, i.e., $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are the maximum likelihood estimates.

The proof of the asymptotic normality of $(\hat{\lambda}_1, \hat{\lambda}_2)$ is a routine application of the theorem on the asymptotic behavior of maximum likelihood estimators. Cramér's conditions [2] for this theorem are satisfied so $\sqrt{N}(\hat{\lambda}_1 - \lambda_1)$ and $\sqrt{N}(\hat{\lambda}_2 - \lambda_2)$ are asymptotically jointly normally distributed with mean (0,0) and covariance matrix equal to the inverse of the information matrix whose elements are given by:

$$E\left[\frac{\partial \log L_{T^*}}{\partial \lambda_1}\right]^2 = \int_0^{T_0} \left(\frac{1}{\lambda_1} - t\right)^2 \lambda_1 e^{-\lambda_1 t} dt + T_0^2 e^{-\lambda_1 T_0},$$

$$= \frac{1}{\lambda_1^2}(1 - e^{-\lambda_1 T_0}),$$

$$E\left[\frac{\partial \log L_{T^*}}{\partial \lambda_2}\right]^2 = \int_{T_0}^{T_1} \left(\frac{1}{\lambda_2} - (t - T_0)\right)^2 \lambda_2 e^{-\lambda_1 T_0 - \lambda_2 (t - T_0)} dt$$

$$+ (T_1 - T_0)^2 e^{-\lambda_1 T_0 - \lambda_2 (T_1 - T_0)},$$

$$= \frac{1}{\lambda_2^2} e^{-\lambda_1 T_0}(1 - e^{-\lambda_2 (T_1 - T_0)}),$$

$$E\left[\frac{\partial \log L_{T^*}}{\partial \lambda_1} \cdot \frac{\partial \log L_{T^*}}{\partial \lambda_2}\right] = -T_0 \left\{ \int_{T_0}^{T_1} \left(\frac{1}{\lambda_2} - (t - T_0)\right) \lambda_2 e^{-\lambda_1 T_0 - \lambda_2 (t - T_0)} dt \right.$$

$$\left. - (T_1 - T_0) e^{-\lambda_1 T_0 - \lambda_2 (T_1 - T_0)} \right\} = 0,$$

where L_{T^*} is the likelihood function of the random variable T^*.

3. Asymptotic normality of $(\hat{\lambda}_1, \hat{\lambda}_2)$; T_0 unknown except for $T_2 \leqq T_0 \leqq T_3$.

For an ordered sample $(t_1, \cdots, t_r, t_{r+1}, \cdots, t_s, t_{s+1}, \cdots, t_n, T_1, \cdots, T_1)$ of N independent observations on the distribution (6) censored at T_1, the proposed estimator of (λ_1, λ_2) is given by $(\hat{\lambda}_1, \hat{\lambda}_2)$ of (15) and (16). The theorem on the asymptotic normality of maximum likelihood estimators cannot be applied directly since $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are not maximum likelihood estimators of λ_1 and λ_2 for the distribution (6) censored at T_1. However, it is possible to apply the theorem indirectly. $\hat{\lambda}_1$ and $\hat{\lambda}_2$ do not involve the actual values of $t_{r+1}, \cdots, t_s$ so, as far as $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are concerned, the sample is $(t_1, \cdots, t_r, \tau, \cdots, \tau, t_{s+1}, \cdots, t_n, T_1, \cdots, T_1)$, where τ in the i-th coordinate denotes that the i-th ordered observation occurred in the interval $[T_2, T_3]$. But this sample can be viewed as a sample of N independent observations from the distribution

$$\begin{aligned}
p_{T^{**}}(t)dt &= \lambda_1 e^{-\lambda_1 t} dt, && 0 \leqq t < T_2, \\
P\{T^{**} = \tau\} &= (1 - p)e^{-\lambda_1 T_2}, && \\
p_{T^{**}}(t)dt &= p\lambda_2 e^{-\lambda_1 T_2 - \lambda_2 (t - T_3)} dt, && T_3 < t < T_1, \\
P\{T^{**} = T_1\} &= pe^{-\lambda_1 T_2 - \lambda_2 (T_1 - T_3)}, &&
\end{aligned} \tag{26}$$

where $0<p<1$. Actually, of course, p is equal to the value

$$p^* = P\{T^* > T_3 \mid T^* \geqq T_2\} = e^{-\lambda_1(T_0-T_2)-\lambda_2(T_3-T_0)}. \tag{27}$$

If for the moment it is forgotten that p is actually given by (27), then (26) is a distribution which depends on three arbitrary, unrelated parameters, λ_1, λ_2, p. Calculation of the likelihood of the sample readily verifies that $(\hat{\lambda}_1, \hat{\lambda}_2)$ is the maximum likelihood estimator of (λ_1, λ_2) for this distribution involving three unknown parameters λ_1, λ_2, p. The conditions of the theorem for asymptotic normality are satisfied so $\sqrt{N}(\hat{\lambda}_1-\lambda_1)$ and $\sqrt{N}(\hat{\lambda}_2-\lambda_2)$ are asymptotically jointly normally distributed with mean (0, 0) and covariance matrix equal to the inverse of the information matrix whose elements are:

$$E\left[\frac{\partial \log L_{T^{**}}}{\partial\lambda_1}\right]^2 = \int_0^{T_2}\left(\frac{1}{\lambda_1} - t\right)^2 \lambda_1 e^{-\lambda_1 t}dt + T_2^2(1 - e^{-\lambda_1 T_2}),$$

$$= \frac{1}{\lambda_1^2}(1 - e^{-\lambda_1 T_2}),$$

$$E\left[\frac{\partial \log L_{T^{**}}}{\partial\lambda_2}\right]^2 = \int_{T_3}^{T_1}\left(\frac{1}{\lambda_2} - (t - T_3)\right)^2 p\lambda_2 e^{-\lambda_1 T_2-\lambda_2(t-T_3)}dt$$

$$+ (T_1 - T_3)^2 pe^{-\lambda_1 T_2-\lambda_2(T_1-T_3)},$$

$$= \frac{pe^{-\lambda_1 T_2}[1 - e^{-\lambda_2(T_1-T_3)}]}{\lambda_2^2},$$

$$E\left[\frac{\partial \log L_{T^{**}}}{\partial\lambda_1}\cdot\frac{\partial \log L_{T^{**}}}{\partial\lambda_2}\right] = -T_2\left\{\int_{T^3}^{T_1}\left(\frac{1}{\lambda_2} - (t - T_3)\right)p\lambda_2 e^{-\lambda_1 T_2-\lambda_2(t-T_3)}dt\right.$$

$$\left. - (T_1 - T_2)pe^{-\lambda_1 T_2-\lambda_2(T_1-T_3)}\right\} = 0.$$

Since, however, the actual value of p is known, there is no harm in using it. Thus, the asymptotic variances of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are

$$\sigma^2_{\hat{\lambda}_1} = \lambda_1^2(1 - e^{-\lambda_1 T_2})^{-1}, \qquad \sigma^2_{\hat{\lambda}_2} = \lambda_2^2[p^* e^{-\lambda_1 T_2}(1 - e^{-\lambda_2(T_1-T_3)})]^{-1}. \tag{28}$$

The results (28) are not serviceable for constructing confidence regions since p^* involves the unknown quantity T_0. But $p^* \geqq \exp(-\lambda_1(T_3-T_2))$ for $\lambda_1 \geqq \lambda_2$, regardless of the value of T_0, so if p^* is replaced by this bound, an upper bound is achieved on the variance $\sigma^2_{\hat{\lambda}_1}$ which permits the construction of confidence regions. The bound $\exp(-\lambda_1(T_3-T_2))$ also has the desirable property that it is a lower bound for p even if the failure rate $\lambda(t)$ decreases in $[T_2, T_3]$ other than by a jump discontinuity. This means that the results of (17) hold for all failure rate models in which $\lambda(t)$ decreases monotonically from λ_1 to λ_2 in the interval $(T_2, T_3]$.

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MATHEMATICAL MODELS FOR RANKING FROM PAIRED COMPARISONS[1]

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Brief mention is made of several models in each of two categories: (I) Each possible ranking of items is assumed to have a "utility" which depends on the expected scores of the items in paired comparisons. In particular, the "worth" of an item may be *defined* in terms of its expected scores in comparisons with others. (II) Each item is *assumed to have an intrinsic worth;* these intrinsic worths determine the expected scores.

A concept, "regularity" is introduced. Under (I), general linear utilities are discussed, and a necessary and sufficient condition is given in order that a linear utility may be regular.

Under (II), a "minimum assumption" model is introduced. Let $e(u, v)$ denote the expected score of an item of worth u when compared with one of worth v. The assumption is: $e(u, v)$ is non-decreasing in u, non-increasing in v. The problem of estimating expected scores in this model is discussed.

1. INTRODUCTION

Most models for ranking from paired comparisons appear to fall in one or the other of two broad categories. Each of these categories is discussed in general, with examples, and a new model is proposed in each.

The typical situation involves a number of comparisons, each between a pair of individuals or items. The result of a single comparison may be a stated preference for one of two foods presented to a judge. It may be the winning of a game by one of two players. It may be that a laboratory animal responds to one of two competing stimuli. In each of these situations, a simple "preference" is expressed for one or the other of two "items." More generally, as a result of a comparison a "score" is assigned to each of the two items. In the simplest situation, that in which a simple preference is expressed, 1 is scored for the preferred item, 0 for the other. The score of an item on a single comparison may be thought of in this case as the "number of preferences expressed for the item." In generalizing so as to admit the possibility of a tie, one may give score $\frac{1}{2}$ to each item in the case of a tie or a draw. In general, if items I_1 and I_2 are to be compared, we suppose scores s_{12} (the score of I_1 in a comparison with I_2) and s_{21} are to be assigned so that

$$s_{12} + s_{21} = 1. \tag{1}$$

(There is thus adopted here a convention different from that of Scheffé [12]; if the score by the above convention is denoted by s and that referred to by Scheffé (reduced by a constant factor so that the maximum score is 1) by S, then $S = 2s - 1$. It is noted further that Scheffé admits the possibility that the order in which the items are presented to a judge may affect the score. This consideration, important in many situations, is neglected here.)

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Each item presented to a judge may have been selected at random from a population of similar items. Each judge may have been selected from a population of judges. The state of a player at a particular game might be considered as randomly chosen from a population of possible states of that player. These remarks are intended to suggest that it is appropriate to consider the score of an item as a random variable. One accordingly postulates the existence of an underlying probability space. Let there be m items, $I_1, I_2, \cdots, I_m$, to be compared in pairs.

Basic assumption

There is a probability space on which are defined random variables S_{ij}, "the score of item I_i in a comparison with item I_j" $(i, j=1, 2, \cdots, m, i \neq j)$ (A1) *such that* $S_{ij}+S_{ji}=1$.

Repeated independent trials of a comparison between specified items may be considered on an appropriate Cartesian product space in the usual way.

Since the range of possible scores will ordinarily be bounded, it is not a serious restriction to assume that moments of all orders of the random variables S_{ij} exist. In particular, the mean will be denoted by e_{ij}:

$$E(S_{ij}) = e_{ij}. \tag{2}$$

The $m \times m$ matrix of which the element in the i-th row and j-th column is e_{ij} is the *matrix of expected scores*, E. From (A1) and (2) it follows that the sum of an element of this matrix and its image in the principal diagonal is 1:

$$e_{ij} + e_{ji} = 1. \tag{3}$$

By convention, $e_{ii}=\frac{1}{2}$, $i=1, 2, \cdots, m$.

2. MODELS FOR RANKING

As a matter of notational convenience, we shall use symbols with superscripts when referring to ranked items, symbols with subscripts when referring to unranked items as originally presented for consideration. Thus $I_1, I_2, \cdots, I_m$ are the "names" of the m items under consideration; $I^1, I^2, \cdots, I^m$ is a specific ranking of them; e^{ij} is the expected score of the i-th ranked item when compared with the j-th, etc.

Given expected scores for items $I_1, I_2, \cdots, I_m$ in pairs, how are the items to be ranked? Approaches to this problem often involve the concept of "worth" of an item, in either of two conceptually distinct ways. In either case, the "worth" of an item is a real number assigned to that item.

(*i*) The worth of an item is *defined* in terms of its expected scores when compared with the other items.

(*ii*) Each item is *assumed to have an intrinsic worth*; these intrinsic worths determine the expected scores.

An assumed intrinsic worth plays a role somewhat analogous to a "main effect" in the analysis of variance (this is most clear in Scheffé's model [12]). To assume they determine completely the matrix of expected scores is analogous to the hypothesis of *no interaction* in the analysis of variance. Such models are

considered in §3. In the present section is considered a general approach to the problem of ranking from given expected scores, including as a special instance the approach by definition of worth (*i*) above.

The basic assumption for this approach is that if pertinent properties of, and relations among, the items were completely known, then it would be possible to say of any two proposed rankings that one is more desirable than the other. For example, if in contests of a certain kind, player A consistently wins over player B who consistently wins over C who in turn wins over A, a committee might be able to agree that the ranking ACB (say), in which no player is ranked more than one place higher than a player who can beat him, is preferable to the ranking ABC. Further, it will be supposed that a numerical scale of desirability could be introduced. That is, given complete information on the items, each ranking would have a numerical "utility" (for the investigator, or for some segment of the community) such that the more desirable of two rankings would have the greater utility. This assumption is labelled (A2):

Each ranking of the items has a "value" or "utility". (A2)

An optimum ranking is then one of maximum utility.

Ranking by worths, if they exist, can be subsumed under (A2). For example, if w^i denotes the worth of item I^i $(i=1, 2, \cdots, m)$ in the ranking $I^1, I^2, \cdots, I^m$, one may take the utility of the ranking to be 1 if $w^1 \geq w^2 \geq \cdots \geq w^m$, and otherwise 0.

We consider first models under (A2) in which the concept of worth does not enter. A further assumption relating the expected scores matrix to the utility of a ranking is:

The utility of a ranking $I^1, I^2, \cdots, I^m$ is a function of the expected scores e^{ij}, $1 \leq i < j \leq m$, which is non-decreasing in each argument. (S1)

(Assumptions relating utilities directly to expected scores will be labelled "S".) The utility of a ranking will also be referred to as the utility of the expected scores matrix corresponding to that ranking.

Model A. A suggestion of Kendall [9], that the number of expressed preferences which are violated be minimized, motivates the following choice of a utility function.

The utility of a ranking $I^1, I^2, \cdots,$ I^m is the sum $\sum_{i=1}^{m} \sum_{j=i+1}^{m} e^{ij}$. (S2)

In order to make clear the relationship between (S2) and the stated motivation, suppose a single comparison is made of each pair of items. Suppose that each comparison results either in a tie or in a preference for one of the two items, and further that each expected score e^{ij} is estimated by the observed score s^{ij}, which is 0, $\frac{1}{2}$, or 1. It is clear that while the estimated e^{ij} appearing in the sum in definition (S2) depend on the ranking, precisely the same number of scores $\frac{1}{2}$ will appear for all rankings. If r scores $\frac{1}{2}$ appear, then $\frac{1}{2}m(m-1)-r$ scores appear which are 0's or 1's. Each 0 appearing in the sum represents an expressed preference which is violated. Since the total number of 0's and 1's ap-

pearing in the sum is the same for all rankings, a ranking which minimizes violated preferences is a ranking for which the (estimated) utility (S2) is maximized. (The same conclusion is reached if each tie counts as half a violated preference.)

Regular utilities. Suppose there are two items I_i and I_j such that $e_{ik} \leq e_{jk}$ for $k = 1, 2, \cdots, m$. A utility will be called *regular* if in each such situation a ranking in which I_j precedes I_i has at least as great utility as that in which I_i and I_j are interchanged. While not every utility satisfying (S1) is regular, one readily verifies that the utility defined by (S2) is regular.

In the absence of utilities for rankings, a method of ranking by worths will be called regular if the utility referred to following (A2) is regular. This is equivalent to requiring that $e_{ik} \leq e_{jk}$ for $k = 1, 2, \cdots, m$ imply that I_j have at least as great worth as I_i.

We turn now to consideration of some models in which worth is determined by expected scores, and in which ranking by worth is regular.

Kendall Model ([9]). In this model the worth of item I_i is defined as

$$w_i = \sum_{j=1}^{m} e_{ij}. \tag{DW1}$$

(Assumptions defining worths will be labelled "*DW*".) One may say of this model too that, from an appropriate point of view, the number of expressed preferences which are violated is minimized. Consider again an experiment in which each item is compared once with each other, and the observed scores $\{s_{ij}\}$ are used to estimate the expected scores $\{e_{ij}\}$. Suppose further that there are no ties, or that a tie is counted as half a violated preference. Then the first item in an optimal ranking under (*DW*1) is that item I_i for which the fewest violated preferences are represented among the scores s_{ij} ($j = 1, 2, \cdots, m$). The first item having been determined, the second is that item I_r among the remaining $m-1$ for which the fewest violated preferences are represented among the scores e_{rj}, $j = 1, 2, \cdots, m$, etc.

With Model *A*, that ranking is optimal for which the total number of violated preferences is minimized. With the Kendall model, the first item is chosen so as to minimize violated preferences, then the second, etc. Thus one might consider Model *A* more appropriate in a situation in which the whole ranking is, so to speak, of uniform interest; while the Kendall model would be more appropriate in the very common situation in which it is of prime importance to choose the highest ranked item correctly, then of secondary importance to choose the second, etc.

If (S2) were to be replaced by "the utility of a ranking $I^1, I^2, \cdots, I^m$ is the sum $\sum_{i=1}^{m} e^{1i}$," one should obtain a model in which *only* the first ranked item is considered of importance. If the utility of a ranking $I^1, I^2, \cdots, I^m$ is taken to be the sum $\sum_{i=1}^{2} \sum_{j=1}^{m} e^{ij}$, one has a model in which it is considered of vital importance that the best *two* be segregated, but not that a choice be made between them, etc. An optimum ranking under the Kendall model (*DW*1) would maximize utility under each of these alternatives to (S2).

Linear utility. These alternatives to (S2) are subsumed under a general model depending on the following assumption:

$$\left.\begin{array}{l}\textit{there are non-negative constants } c^{ij}(i, j = 1, 2, \cdots, m) \textit{ such that the}\\ \textit{utility of the ranking } I^1, I^2, \cdots, \quad I^m \textit{ is the sum } \sum_{i=1}^{m}\sum_{j=1}^{m} c^{ij}e^{ij}.\end{array}\right\} \quad \text{(S3)}$$

One obtains Model A on setting $c^{ij}=1$ for $1\leq i<j\leq m$, $c^{ij}=0$ otherwise, and the two above by setting respectively $c^{ij}=1$, $c^{ij}=0$ for $i\geq 2$, and $c^{ij}=1$ for $i=1, 2$, $c^{ij}=0$ for $i>2$. The Kendall model optimal ranking will maximize utility under (S3) if $c^{ij}=c^i$ $(k, j=1, 2, \cdots, m)$, where $c^1, c^2, \cdots, c^m$ satisfy the condition $c^1\geq c^2\geq \cdots \geq c^m$. One might think of the relative value of c^i as specifying the importance attached to correct determination of the i-th ranked item.

Multiplying each c^{ij} by a constant has the effect of multiplying the utility by that constant. Further, since $e^{ij}+e^{ji}=1$, if a is any number and if, for a particular pair (i, j), c^{ij} and c^{ji} are replaced by $c^{ij}+a$ and $c^{ji}+a$ respectively, the utility is simply increased by a. Suppose M is the largest of the sums $c^{ij}+c^{ji}$. Set

$$\gamma^{ij} = \frac{1}{M}\left[c^{ij} + \tfrac{1}{2}(M - c^{ij} - c^{ji})\right](i, j = 1, 2, \cdots, m). \tag{4}$$

The utility calculated using the $\{\gamma^{ij}\}$ in place of the $\{c^{ij}\}$ is a linear function of the utility using the $\{c^{ij}\}$. The utility

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \gamma^{ij}e^{ij}$$

will be called the *normalized* utility. One has

$$0 \leq \gamma^{ij} \leq 1, \qquad \gamma^{ij} + \gamma^{ji} = 1, \qquad (i, j = 1, 2, \cdots, m). \tag{5}$$

It is shown in the appendix (Theorem 1) that a linear utility is regular if and only if after normalization one has

$$\gamma^{ik} \geq \gamma^{jk}, \qquad \gamma^{ki} \leq \gamma^{kj} \quad \text{for} \quad i < j, \qquad k = 1, 2, \cdots, m. \tag{6}$$

Under these conditions the utility satisfies (S1) also (Theorem 2).

Kendall-Wei Model ([9]). In this model the worth w_i of the i-th item is a function of the expected scores e_{ij} $(j=1, 2, \cdots, m)$ satisfying

$$\lambda w_i = \sum_{j=1}^{m} e_{ij}w_j, \qquad i = 1, 2, \cdots, m, \tag{DW2}$$

for some positive constant λ. Thus a high expected score when compared with a worthy item contributes more to worth than a high expected score when compared with a less worthy item. It is clear that λ must be a characteristic value, and $w=(w_1, w_2, \cdots, w_m)$ a corresponding characteristic vector, of the expected scores matrix. One observes that ranking by worths determined by (*DW*2) is regular.

Dominance ranking method. G. L. Thompson [13] has studied and generalized

the Kendall-Wei method, introducing a "dominance ranking method." Let D denote a matrix identical with the expected scores matrix except that 0's appear on the main diagonal; e a vector each of whose components is 1, and r a positive number less than the reciprocal of the principal characteristic value, λ, of E. Then a worth vector $w=(w_1, w_2, \cdots, w_m)$ is defined by

$$w = (D + rD^2 + r^2D^3 + \cdots)e.$$

The definition may be extended to the case $r=1/\lambda$, when Thompson finds that the dominance ranking method coincides with that of Kendall and Wei.

Clearly the dominance ranking method is also regular.

3. INTRINSIC WORTH MODELS

In such models one assumes *a priori* that the items have intrinsic worths which determine the expected scores matrix. Such worths are analogous to "main effects" in the analysis of variance, and the assumption is analogous to the assumption of "no interaction." The following property of intrinsic worth seems a desirable one.

The expected score of an item of worth u when compared with one of worth v is a function $e(u, v)$ such that $u>v$ implies $e(u, v)\geq e(v, u)$, or, equivalently, $e(u, v)\geq\frac{1}{2}$. (*IW*1)

(Assumptions concerning intrinsic worths will be labelled "*IW*".) (Logical relations among this assumption and others are studied in ref. [1].) It is shown in the appendix (Theorem 3) that a necessary and sufficient condition that ranking by intrinsic worth be regular is that (*IW*1) hold. Intrinsic worth satisfying (*IW*1) will be called *weakly regular* (as contrasted with strongly regular intrinsic worth, defined below).

A stronger assumption is the following.

The expected score of an item of worth u when compared with an item of worth v is a function $e(u, v)$ which is non-decreasing in u, non-increasing in v. (*IW*2)

(Note that one of the requirements is redundant: if $e(u, v)$ is non-decreasing in u it must be non-increasing in v, and conversely, since $e(u, v)+e(v, u)=1$.) It is shown in the appendix (Theorem 4) that a necessary and sufficient condition that ranking by intrinsic worth coincide with the optimal ranking by every regular utility is that (*IW*2) hold. An intrinsic worth satisfying (*IW*2) will be called *strongly regular*.

Condition (*IW*2) appears as Axiom 2 in Appendix I of [10].

A "quadruple condition" appears in references [1] and in [8]: $e_{hi}\geq e_{jk}$ implies $e_{hj}\geq e_{ik}$. It is easy to verify that in the presence of (*IW*1) this condition implies (*IW*2).

Several models specifying the form of the function $e(u, v)$ satisfying (*IW*2) are mentioned briefly below. In some situations, however, it is not easy to justify, *a priori*, a particular form for $e(u, v)$. For such situations a *basic no-interaction model* is proposed, in which it is assumed only that a function $e(u, v)$ satisfying (*IW*2) exists, without specifying its form *a priori*. Discussion

of this "minimal" model is deferred till after brief consideration of some others. Each satisfies the following assumption with a specified function H:

$$e(u, v) = H(u - v),$$

where H is the distribution function of a symmetric random variable. That is, H is a non-decreasing function such that $H(-\infty)=0, H(\infty)=1$, and $H(-x)+H(x)\equiv 1$. (*IW*3)

Condition (*IW*3) implies (*IW*2). Theorem 6 in the appendix states that (*IW*3) is essentially stronger if $m>3$; that is, a situation is exhibited in which (*IW*2) holds but not (*IW*3). This has also been proved by Block and Marschak [1].

Scheffé Model. In the discussion of the Scheffé model, attention is restricted to the case in which the order in which items are presented to a judge does not affect the score. Making allowance for the difference in scoring method, one finds that under the "hypothesis of subtractivity" the Scheffé model is specified by taking for H the distribution function of the uniform or rectangular distribution over $(-\frac{1}{2}, \frac{1}{2})$:

$$H(x) = \tfrac{1}{2} + x, \qquad -\tfrac{1}{2} < x < \tfrac{1}{2},$$

or

$$e(u, v) = \tfrac{1}{2} + (u - v). \qquad (IW4)$$

When the "hypothesis of subtractivity" fails, Scheffé defines "deviations from subtractivity" analogous to interactions:

$$\gamma_{ij} = e_{ij} - [\tfrac{1}{2} + (w_i - w_j)],$$

where w_i and w_j are the worths of items I_i and I_j respectively.

A property of (*IW*4) which may be disadvantageous for some purposes is that it does not allow for great variations in worth. That is to say that if, for example, e_{12} is near 1, then e_{23} can not be. For $-\frac{1}{2} \leq \min_{i,j}(w_i - w_j) \leq \max_{i,j}(w_i - w_j) \leq \frac{1}{2}$, and if $w_1 > w_2 > w_3$ with $w_1 - w_2$ near $\frac{1}{2}$ then $0 \leq w_2 - w_3 = (w_1 - w_3) - (w_1 - w_2) \leq \frac{1}{2} - (w_1 - w_2)$ so that $w_2 - w_3$ must be near 0 and e_{23} near $\frac{1}{2}$. However, in many or most situations the items to be compared will be of comparable worth. One might in such situations expect the advantages of the linearity of the model and its close relationship to the analysis of variance to make it very useful.

It is interesting to note that intrinsic worth under (*IW*4) is consistent with worth as defined by (*DW*1). That is to say, if w_i represents the worth of I_i under (*IW*4) then w_i may be chosen so as to be given also, except for a constant multiplier, by (*DW*1):

$$w_i = \frac{1}{m} \sum_{j=1}^{m} e_{ij}.$$

To see this, compute the right hand member, assuming (*IW*4):

$$\frac{1}{m} \sum_{j=1}^{m} e_{ij} = \frac{1}{m} \sum_{j=1}^{m} [\tfrac{1}{2} + (w_i - w_j)] = w_i + \frac{1}{m} \sum_{j=1}^{m} (\tfrac{1}{2} - w_j).$$

Noting that the probabilities e_{ij} depend only on the differences $w_i - w_j$, one may normalize by setting

$$\frac{1}{m}\sum_{j=1}^{m}(\tfrac{1}{2} - w_j) = 0$$

(this is also the normalization used by Scheffé). Then

$$w_i = \frac{1}{m}\sum_{j=1}^{m} e_{ij}.$$

Bradley-Terry Model [2]. The Bradley-Terry model is based on the intuitively appealing idea that the ratio e_{ij}/e_{ji} should be that of the worths w_i/w_j:

$$\left.\begin{aligned} e_{ij}/e_{ji} &= w_i/w_j; \quad \textit{or equivalently,} \\ e_{ij} &= w_j/(w_i + w_j). \end{aligned}\right\} \tag{IW5}$$

After a simple change of variable, $w^* = \ln w$, intrinsic worths under (*IW*5) satisfy also (*IW*3) with [3]

$$H(x) = \frac{1}{4}\int_{-\infty}^{x} \operatorname{sech}^2 (y/2)dy.$$

Thus

$$e(u, v) = u/(u + v) = H(u^* - v^*) \tag{IW6}$$

where $u^* = \ln u$, $v^* = \ln v$.

Normal Distribution Model.[2] This model (cf. [11]) satisfies (*IW*3) with

$$H(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-t^2/2)dt. \tag{IW7}$$

Assumption (*IW*3) is a consequence of other assumptions ([7], p. 175) more directly related to some situations where the appropriateness of the model might be considered. In taste tests, for example, or other situations where a subject is presented a stimulus, one might assume that

$$\left.\begin{aligned} &\textit{the response } X \textit{ of a subject chosen at random from the population of} \\ &\textit{subjects to a stimulus has a distribution with distribution function} \\ &F(x-S), \textit{ where } S \textit{ is a value associated with the stimulus. His response} \\ &Y \textit{ to another is independent of } X \textit{ and has distribution function} \\ &F(y-T), \textit{ where } T \textit{ is a value associated with the other stimulus.} \end{aligned}\right\} \tag{IW8}$$

It is assumed he will prefer that with value T if $Y > X$. The probability of this is readily seen to be $H(T-S)$, where H is the distribution function of $U-V$, where U and V are independent random variables with distribution function F. Thus (*IW*8) implies (*IW*3); but not, of course, conversely, since not every distribution function H is of the form required by (*IW*8).

Mosteller [11] points out that if F is the normal distribution, it is not neces-

[2] Professor Jacob Marschak has kindly informed the writer that this model should be attributed to Fechner: Gustav Theodor Fechner, *Elemente der Psychophysik*, 1859 (cf. especially pp. 70–103 of the 1889 edition).

sary to assume that X and Y are independent, only that their correlation is independent of S and T, in order to arrive at $(IW7)$.

Basic no-interaction model. This model is based on $(IW2)$ alone: there exist intrinsic worths and a function $e(u, v)$ giving the expected score of an item of worth u in comparison with one of worth v, which is non-decreasing in u, non-increasing in v. It follows that, given m items,

$$\text{there exists a ranking } I^1, I^2, \cdots, I^m \text{ such that } e^{ij} \leq e^{ik},\ e^{jk} \leq e^{ik} \text{ for } 1 \leq i < j < k \leq m; \tag{IW9}$$

namely, ranking by worth. It is shown in the appendix (Theorem 5) that the converse is also true: given m items and a ranking satisfying $(IW9)$, there can be found worths $w^1, w^2, \cdots, w^m$ and a function $e(u, v)$ satisfying $(IW2)$ such that $e(w^i, w^j) = e^{ij}$ $(i, j = 1, 2, \cdots, m)$. It is further observed that total scores (Kendall model) $s^i = \sum_{j=1}^{m} e^{ij}$ will serve as worths w^i, as will also Kendall-Wei worths (Corollary 1 to Theorem 5).

4. ESTIMATION WITH THE BASIC NO-INTERACTION MODEL

The remarks above have been concerned chiefly with the matrix E of expected scores, and little mention has been made of the distributions of the scores themselves. If a simple preference is expressed at a single comparison between I_i and I_j (a tie assumed impossible), the score S_{ij} has an elementary binomial distribution: $S_{ij} = 1$ with probability e_{ij}, $S_{ij} = 0$ with probability $e_{ji} = 1 - e_{ij}$. The general discussion admits of arbitrary distributions, subject to the existence of expectations. Let k_{ij} be 0 or a positive integer, and let k_{ij} trials be made of the comparison between I_i and I_j $(i, j = 1, 2, \cdots, m)$. If $S_{ij\nu}$ is the score of I_i in the ν-th trial, let

$$S_{ij} = \frac{1}{k_{ij}} \sum_{\nu=1}^{k_{ij}} S_{ij\nu}$$

denote the arithmetical average of the k_{ij} scores. If the distributions of the scores depend on certain parameters, among which are the expected scores, one may, given a specified ranking, choose the parameters so as to maximize the joint density function of the scores, subject to $(IW9)$. The ranking for which this maximum value is largest is then the maximum likelihood (M.L.) ranking. Suppose it is known that the scores $\{S^{ij\nu}\}$ have distributions all belonging to the same exponential family (cf. appendix), with means $\{e^{ij}\}$. In particular, they may all have the elementary binomial distribution, in which e^{ij} is the probability that the i-th ranked item will be preferred to the j-th ranked. Given a ranking, and given observed values $s^{ij\nu}$, the problem of maximum likelihood estimation of the e^{ij} subject to $(IW9)$ is essentially a special instance of the problem solved in ref. [4] (cf. [5]). Consider a square array of points (i, j), i running from 1 to m from top to bottom, j running from 1 to m from left to right, as rows and columns are numbered in the scores matrix. However, the points (i, i) $(i = 1, 2, \cdots, m)$ are to be missing from this array, as are also all points (i, j) for which $k^{ij} = 0$. In the terminology of [4] an *upper interval* consists of a point (i_0, j_0) together with all points (i, j) such that $i \leq i_0$, $j \geq j_0$. An

upper layer is a union of upper intervals. A *lower interval* consists of a point (i_0, j_0) together with all points (i, j) such that $i \geq i_0$, $j \leq j_0$. A *lower layer* is a union of lower intervals. A *layer* is the intersection of an upper layer with a lower layer. The *average score* over a layer is the sum of the scores S^{ij} from all observations at points (i, j) in the layer divided by the total number of observations at these points.

The solution as given in ref. [5], §5, may be described as follows. Consider the upper layer over which the average score is largest. If there are two or more upper layers having the same largest average score, so does their union, which is the *maximal upper layer, U^1, over which the average score is maximal.* For each point (i, j) in U^1, the maximum likelihood estimate (MLE) $\hat{e}^{ij}$ of e^{ij} is precisely this maximal average score. Consider now those layers which are intersections of upper layers with the complement of U^1. Among these layers again choose the maximal layer, U^2, over which the average score is maximal. At each point of this layer the *MLE* $\hat{e}^{ij}$ of e^{ij} is again this maximal average score. Consider next layers which are intersections of upper layers with the complement of $U^1 \cup U^2$, etc.

This method of solution is illustrated in the example below, and in the appendix, Theorem 7.

One obtains the same solution starting at the lower left instead of at the upper right. That is, one may choose first the maximal lower layer, L^1, over which the average score is minimal. The *MLE* $\hat{e}^{ij}$ of e^{ij} for each point (i, j) in L^1 is precisely the average score over L^1. Consider next layers which are intersections of lower layers with the complement of L^1, etc.

Since $S^{ij\nu} + S^{ji\nu} = 1$ $(\nu = 1, 2, \cdots, k^{ij}; i, j = 1, 2, \cdots, m, i \neq j)$, the layers L^1 and U^1 are symmetrically situated with respect to the line $i = j$ (the principal diagonal) as are also the layers L^2 and U^2, etc. Moreover the sum of the average scores over L^1 and U^1 is 1, the sum of the average scores over L^2 and U^2 is 1, etc. Thus $\hat{e}^{ij} + \hat{e}^{ji} = 1$ $(i \neq j, i, j = 1, 2, \cdots, m)$. If a layer on which the *MLE* is constant contains a position (i, i) (a diagonal position), the average score over that layer must be $\frac{1}{2}$. In any event one may, without violating (IW9), adopt the convention $\hat{e}^{ii} = \frac{1}{2}$.

It is true that in ref. [4] (cf. [5]) the random variables $S^{ij\nu}$ at different points (i, j) are assumed independent, while here $S^{ji\nu} = 1 - S^{ij\nu}$. The solution described can be justified, however, on the basis of consideration of observations at points above the principal diagonal $(i < j)$.

For score matrices of moderate size, these calculations are rapid and easy to carry out, given an assumed ranking. One can then calculate the value of the joint density function for each ranking, and choose that ranking for which the joint density function assumes its largest value.

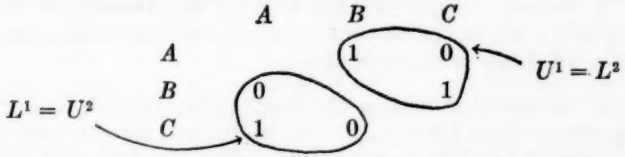
It is observed, [6], p. 444, that when only order restrictions are made on the parameters (expected scores), the MLE's are independent of the particular exponential family to which the distributions of the scores all belong. The order restrictions there contemplated are of the form: "$e^{ij} \leq e^{rs}$ and $e^{uv} \leq e^{wz}$ and $\cdots$." The present situation is an instance, provided a ranking is specified; but before a ranking is specified, the order restrictions on the e_{ij} here are of the form: "$e_{ij} \leq e_{rs}$ and $e_{uv} \leq e_{wz}$ and $\cdots$, *or* $e_{ab} \leq e_{cd}$ and $e_{rs} \leq e_{jk}$ and $\cdots$," and are thus of a kind to which the assertion in ref. [6] does not apply. Indeed, it is

shown in the appendix (Theorem 7) that the *ML* ranking may depend on the particular exponential family to which the distributions of all scores belong. In particular, the least squares estimates need not coincide with those obtained under the assumption that all scores have the elementary binomial distribution.

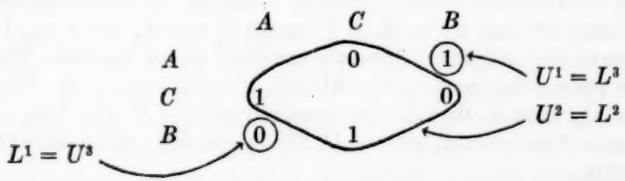
The following property of estimation of ranking under the basic no-interaction model, related to regularity, will often make it unnecessary to consider the maximum value of the joint density function for all possible rankings. It is proved in the appendix (Theorem 8) that if the distributions of all scores belong to the same exponential family, if $k_{ij}=k>0$ $(i, j=1, 2, \cdots, m, i\neq j)$, and if for some pair i, j we have $s_{iv}\geq s_{jv}$ for $v=1, 2, \cdots, m$, then I_i precedes I_j in a *ML* ranking. As a corollary, if the scores matrix is reducible, items corresponding to the rows distringuished by 0's rank below the others in a *ML* ranking. (The scores matrix is reducible if there are a positive integer r less than m and a ranking such that in the corresponding scores matrix an $r\times(m-r)$ matrix of 0's stands in the lower left corner and an $(m-r)\times r$ matrix of 1's in the upper right corner.)

The following simple example illustrates the method and points up the difference between (*IW*2) and (*IW*3). Of three items A, B, C let A be preferred to B, B to C, and C to A. (Each score is assumed to have the elementary binomial distribution.) Maximum likelihood estimates of the worths of A, B and C under (*IW*3) for a specified H (or simply assuming (*IW*3) satisfied for some unspecified function H) would be equal, and each of the six possible rankings would be a *ML* ranking.

For the ranking *ABC*, the observed scores matrix is



One has $k^{ij}=1$ for $i, j=1, 2, 3, i\neq j$. *MLE*'s under (*IW*9) for the ranking *ABC* are $\hat{p}_{AB}=\hat{p}_{AC}=\hat{p}_{BC}=2/3$, $\hat{p}_{BA}=\hat{p}_{CA}=\hat{p}_{CB}=1/3$. The corresponding value of the joint frequency function is $[(2/3)^1(1/3)^0]^4[(2/3)^0(1/3)^1]^2=2^4/3^6$. For the ranking *ACB*



$\hat{p}_{AC}=\hat{p}_{CA}=\hat{p}_{CB}=\hat{p}_{BC}=\frac{1}{2}$, $\hat{p}_{AB}=1$, $\hat{p}_{BA}=0$. The corresponding value of the joint frequency function is $[(\frac{1}{2})^0(\frac{1}{2})^1]^4[1^10^0]^2=1/2^4>2^4/3^6$. Thus *ML* rankings under (*IW*2) are *ACB*, *CBA*, and *BAC*. In none of these rankings is an item ranked more than one place higher than an item which was preferred to it.

5. MULTIPLE COMPARISONS

Much of the above discussion can be modified so as to apply to situations involving multiple comparisons. Discussion here is limited to brief mention of

the adaptation of the basic no-interaction model to a situation where each judge is asked to choose the "best" of three items presented to him. Let p_{ijk} denote the probability that I_i will be preferred to both I_j and I_k. Then

$$p_{ikj} = p_{ijk} \quad \text{and} \quad p_{ijk} + p_{jik} + p_{kij} = 1.$$

In addition, in the basic no-interaction model,

$$p_{ijk} \textit{ is non-increasing in } i, \textit{ non-decreasing in } j \textit{ and in } k. \qquad (IW10)$$

The problem of estimating the p^{ijk} subject to ($IW10$), given a ranking, is again essentially a special instance of the problem solved in ref. [4] (cf. [5]), involving a three-dimensional array.

APPENDIX

THEOREM 1. *A necessary and sufficient condition that a linear utility be regular is that after normalization* $\gamma^{ik} \geq \gamma^{jk}$, $\gamma^{ki} \leq \gamma^{kj}$ for $i<j$, $k=1, 2, \cdots, m$. (Cf. (§3).)

PROOF OF SUFFICIENCY. The change in the expected scores matrix E brought about by an interchange of two items in a ranking can be thought of as accomplished by successive interchanges of elements of E: two in a common row, two in a common column, or two which are images of each other in the principal diagonal. Note that if there are two rows of E such that each element in one row is not greater than the element in the same column of the other row, then the diagonally opposite elements in these rows satisfy the same inequality (the one in the row of smaller elements being not greater than $\frac{1}{2}$, the other not smaller than $\frac{1}{2}$). It therefore suffices to consider a single interchange of two elements of E. Suppose $i \leq j$, $r \geq s$; then $\gamma^{is} \leq \gamma^{ir}$. Suppose $e^{ir} \geq e^{is}$. Then $\gamma^{ir}e^{ir} + \gamma^{is}e^{is} \geq \gamma^{ir}e^{is} + \gamma^{is}e^{ir}$. Repeated application of this observation completes the proof of the sufficiency of the condition.

PROOF OF NECESSITY. Assuming the linear utility regular, it is shown first that if $i<j$ then $\gamma^{ir} \geq \gamma^{jr}$ for $r=1, 2, \cdots, m$. Suppose the contrary, that there exist i, j, r with $i<j$ such that $\gamma^{ir}<\gamma^{jr}$. Since $\gamma^{\lambda\mu}+\gamma^{\mu\lambda}=1$ ($\lambda, \mu=1, 2, \cdots, m$), it follows that $\gamma^{ri}>\gamma^{rj}$.

Suppose first that $r \neq i$, $r \neq j$. Consider the expected scores matrix E in which $e^{\lambda\mu}=\frac{1}{2}$ for $\lambda, \mu=1, 2, \cdots, m$, except that $e^{ir}=\frac{1}{2}+h$, $e^{ri}=\frac{1}{2}-h$ ($0<h\leq\frac{1}{2}$). Consider also the expected scores matrix E_1 obtained by interchanging items I^i and I^j. Every element $e_1^{\lambda\mu}$ in E_1 is $\frac{1}{2}$, except $e_1^{jr}=\frac{1}{2}+h$, $e_1^{rj}=\frac{1}{2}-h$. Let $U(E)$, $U(E_1)$ denote the utilities of the two expected scores matrices. Then $U(E_1) - U(E) = \gamma^{jr}h - \gamma^{ri}h - (\gamma^{ir}h - \gamma^{ri}h) = h[(\gamma^{jr}-\gamma^{ir}) + (\gamma^{ri}-\gamma^{rj})] > 0$. Thus E_1 has greater utility than E, though each expected score of I^i is at least as large as the corresponding expected score of I^j. This contradicts the assumed regularity of the utility.

If $r=i$ or $r=j$ one arrives similarly at a contradiction, starting with an expected scores matrix E in which each element is $\frac{1}{2}$ except that $e^{ij}=\frac{1}{2}+h$, $e^{ji}=\frac{1}{2}-h$.

Since $\gamma^{\lambda\mu}+\gamma^{\mu\lambda}=1$ ($\lambda, \mu=1, 2, \cdots, m$), it follows from the inequality $\gamma^{ir} \geq \gamma^{jr}$ ($r=1, 2, \cdots, m$) that also $\gamma^{ri} \leq \gamma^{rj}$ ($r=1, 2, \cdots, m$) if $i<j$. This completes the proof of the theorem.

THEOREM 2. *Under the hypotheses of Theorem* 1, *the linear utility satisfies* (S1).

PROOF. One has

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\gamma^{ij}e^{ij} = \sum_{i=1}^{m}\sum_{j=i+1}^{m} e^{ij}(\gamma^{ij}-\gamma^{ji}) + \sum_{i=1}^{m}\sum_{j=i+1}^{m}\gamma^{ji} + \frac{1}{2}\sum_{i=1}^{m}\gamma_j{}^{ii},$$

so that the utility is a non-decreasing function of the e^{ij}, $j>i$, if and only if $\gamma^{ij}\geq\gamma^{ji}$ for $j>i$. This condition on the γ^{ij} is clearly implied by the hypotheses of Theorem 1 (set $k=j$).

THEOREM 3. *A necessary and sufficient condition that ranking by intrinsic worth be regular is that* (*IW*1) *hold.*

PROOF OF NECESSITY. Suppose the contrary, that ranking by worth is regular, but that there exist u, v such that $u>v$, $e(u, v)<\frac{1}{2}$. Let I_1 have worth u, $I_2, \cdots, I_m$ have worth v. Then $e_{11}=\frac{1}{2}$, $e_{1k}<\frac{1}{2}$, $k=2, 3, \cdots, m$, while $e_{21}>\frac{1}{2}$, $e_{2k}=\frac{1}{2}$, $k=2, \cdots, m$. Since $u>v$, I_1 ranks above I_2, contradicting regularity.

PROOF OF SUFFICIENCY. Suppose (*IW*1) holds, and suppose $e_{ik}\geq e_{jk}$, $k=1, 2, \cdots, m$. Then *a fortiori* $e_{ij}\geq e_{jj}=\frac{1}{2}$. That is, $e(w_i, w_j)\geq\frac{1}{2}$, hence $w_i\geq w_j$. Thus the ranking is regular.

THEOREM 4. *A necessary and sufficient condition that the ranking by intrinsic worth coincide with the optimal ranking by every regular utility is that* $e(u, v)$ *satisfy* (*IW*2).

PROOF. The sufficiency is evident. To prove the necessity, suppose the contrary, that ranking by intrinsic worth coincides with that by every regular utility, but that there exist u, v, r such that $u>r$, $e(u, v)<e(r, v)$. Consider first the case in which $u>r>v$. Let I_1 have worth u, I_2 worth r, $I_3, I_4, \cdots, I_m$ each worth v. Then $e_{1k}=e(u, v)$ for $k=3, 4, \cdots, m$, and $e_{2k}=e(r, v)>e(u, v)=e_{1k}$ for $k=3, 4, \cdots, m$. Choose c^{ij} so that $c^{1k}=1$ for $k=3, \cdots, m$, otherwise $c^{ij}=0$. One verifies that the utility $\sum_i\sum_j c^{ij}e^{ij}$ is regular (Theorem 1), but in an optimal ranking by this utility I_2 precedes I_1. The proof for the other cases is similar.

THEOREM 5. *If* (*IW*9) *holds then there exist worths* $w^1, w^2, \cdots, w^m$ *and a function* $e(u, v)$ *satisfying* (*IW*2) *such that* $e(w^i, w^j)=e^{ij}$ $(i, j=1, 2, \cdots, m)$.

PROOF. Given (*IW*9), choose positive numbers $w^1, w^2, \cdots, w^m$ such that $w^1>w^2>\cdots>w^m$. At points (w^i, w^j), $(i\leq j$, in the u, v plane define $e(w^i, w^j)=e^{ij}$. At other points in the region $w^1\geq u\geq v\geq w^m$ define $e(u, v)$ so as to be locally planar; that is, so that any triple of adjacent points (in three-space) (w^i, w^j, e^{ij}), $(w^{i-1}, w^j, e^{i-1,j})$, $(w^i, w^{j+1}, e^{i,j+1})$ or (w^i, w^j, e^{ij}), $(w^i, w^{j+1}, e^{i,j+1})$, $(w^{i+1}, w^{j+1}, e^{i+1,j+1})$ determine a triangle which, with its interior, lies in the graph of $e(u, v)$. These triangles make up the graph of $e(u, v)$ for $w^1\geq u\geq v\geq w^m$. For $w^1\geq v\geq u\geq w^m$ one defines $e(u, v)=1-e(v, u)$, and verifies that e so defined is non-decreasing in u, non-increasing in v.

COROLLARY 1. *Kendall worths or Kendall-Wei worths will serve as worths if* (*IW*9) *holds.*

It is clear in the proof that any determinations of the worths satisfying $w^1 > w^2 > \cdots > w^m$ will serve, the function $e(u, v)$ being defined after worths have been so chosen. Kendall worths, $w^i = \sum_j e^{ij}$, satisfy the required inequalities, given ($IW9$). So also do Kendall-Wei worths, for if λ is the largest positive characteristic value of E and $x = (x^1, \cdots, x^m)$ a corresponding non-negative vector, then $\lambda x^{i+1} = \sum_j e^{i+1,j} x^j \leq \sum_j e^{ij} x^j = \lambda x^i$.

THEOREM 6. *Condition* ($IW2$) (*or equivalently* ($IW9$)) *does not imply* ($IW3$). (This is proved by Block and Marschak [1].)

PROOF. In the following situation ($IW9$) holds but not ($IW3$). Set $m = 4$, and let e^{ij} $(i, j = 1, 2, 3, 4)$ be expected scores such that $e^{12} < e^{23} < e^{34} < e^{24} < e^{13} < e^{14}$. Then ($IW9$) is satisfied. But if there exists a distribution function H such that ($IW3$) holds, then $e^{12} < e^{23}$ implies $H(w^1 - w^2) < H(w^2 - w^3)$ which implies $w^1 - w^2 < w^2 - w^3$ or $2w^2 - w^3 > w^1$. Similarly $w^2 - w^3 < w^3 - w^4$, and $2w^3 - w^2 > w^4$, whence $w^2 + w^3 > w^1 + w^4$. But also $w^2 - w^4 < w^1 - w^3$, a contradiction. One obtains easily a similar counterexample for arbitrary $m > 3$.

Theorems 7 and 8 require certain properties of exponential families (for references cf. [6]). Let X_0 be a random variable whose moment-generating function exists on an open interval containing the origin (in particular, any bounded random variable), and let $F_0(x)$ be its distribution function. Set

$$\phi(\tau) = \int_{(-\infty,\infty)} e^{x\tau} dF_0(x), \qquad \Theta(\tau) = \log \phi(\tau).$$

Then

$$F(x; \tau) = \int_{(-\infty,x)} \exp\left[u\tau - \Theta(\tau)\right] dF_0(u)$$

is the member corresponding to the parameter value τ of the exponential family of distribution functions determined by $F_0(x)$. Unless X_0 is degenerate, $\Theta(\tau)$ is strictly convex, its derivative $\theta(\tau)$ strictly increasing. If X_τ has distribution function $F(x; \tau)$, then $E(X_\tau) = \theta(\tau)$. Let $\tau(\theta)$ be the inverse function of $\theta(\tau)$, and define

$$T(\theta) = \int_{\theta_0}^{\theta} \tau(v) dv,$$

where $\theta_0 = \theta(0)$; $T(\theta)$ is also strictly convex. Suppose that the distributions of independent random variables $X_{i\nu}$, $\nu = 1, 2, \cdots, k_i$, $i = 1, 2, \cdots, r$ all belong to the same exponential family; that $E(X_{i\nu}) = \theta_i$, $\nu = 1, 2, \cdots, k_i$, $i = 1, 2, \cdots, r$; that $x_{i\nu}$ represents an observed value of $X_{i\nu}$; and that

$$\bar{x}_i = \frac{1}{k_i} \sum_{\nu=1}^{k_i} x_{i\nu}$$

with respect to an appropriate measure independent of the parameter values θ_i, the logarithm of the joint density function of the random variables $X_{i\nu}$ at the observed values is given by

$$\sum_{i=1}^{r} k_i \{\bar{x}_i \tau(\theta_i) - \Theta[\tau(\theta_i)]\}.$$

The *MLE* of $\{\theta_i\}$ therefore minimizes

$$\sum_{i=1}^{r} k_i\{T(\bar{x}_i) + \Theta[\tau(\theta_i) - \bar{x}_i\tau(\theta_i)\}. \tag{7}$$

Given a ranking and a corresponding observed scores matrix, let $L^1, L^2, \cdots, L^r$ be regions in which the *MLE* of the e^{ij} are the constants $s^1, s^2, \cdots, s^r$ respectively. Let n^ν be the total number of observations at a point in L^ν, $\nu = 1, 2, \cdots, r$. Then the value of (7) for the *MLE*'s $\{\hat{e}^{ij}\}$ of the $\{e^{ij}\}$ is

$$\sum_{i,j} k^{ij}\{T(s^{ij}) + \Theta[\tau(\hat{e}^{ij})] - s^{ij}\tau(\hat{e}^{ij})\}. \tag{8}$$

From the definitions of the functions involved one has the identity

$$T(\theta) + \Theta[\tau(\theta)] - \theta\tau(\theta) \equiv 0.$$

Hence

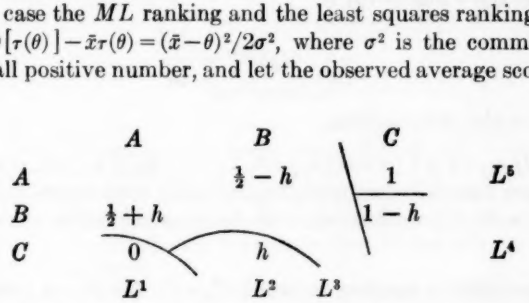
$$\Theta[\tau(s^\nu)] - s^\nu\tau(s^\nu) = -T(s^\nu), \quad \nu = 1, 2, \cdots, r.$$

Since $\hat{e}^{ij} = s^\nu$ for (i, j) in L^ν and since s^ν is the average of the s^{ij} over L^ν, (8) is equal to

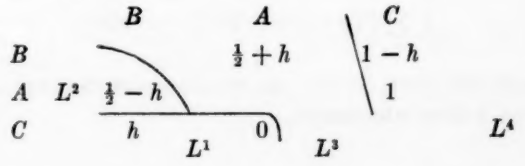
$$\sum_{\nu=1}^{r}\Big[\sum_{ij\in L^\nu} k^{ij}T(s^{ij}) - n^\nu T(s^\nu)\Big]. \tag{9}$$

THEOREM 7. *If the random variables $S^{ij\nu}$ all have distributions belonging to the same exponential family, the maximum likelihood ranking under the basic no-interaction model may depend on the exponential family.* (Cf. Section 4.)

PROOF. An example is given of an observed scores matrix in which one ranking is the *ML* ranking if the $S^{ij\nu}$ all have elementary binomial distributions, while another is the *ML* ranking if the $X^{ij\nu}$ are all normal with a common variance. (In the latter case the *ML* ranking and the least squares ranking coincide, for then $T(\bar{x}) + \Theta[\tau(\theta)] - \bar{x}\tau(\theta) = (\bar{x}-\theta)^2/2\sigma^2$, where σ^2 is the common variance.) Let h be a small positive number, and let the observed average scores matrix be as follows



for ranking ABC, k observations having been made on each pair. For ranking BAC, one has the average scores matrix



Under ABC the MLE's are $\hat{e}_{AB}=\hat{e}_{BA}=\frac{1}{2}$, $\hat{e}_{AC}=1-\hat{e}_{CA}=1$, $\hat{e}_{BC}=1-\hat{e}_{CB}=1-h$. Under BAC the MLE's are $\hat{e}_{BA}=1-\hat{e}_{AB}=\frac{1}{2}+h$, $\hat{e}_{BC}=\hat{e}_{AC}=1-\hat{e}_{CA}=1-\hat{e}_{CB}=1-\frac{1}{2}h$. For the elementary binomial distribution we have

$$T(\theta) = \theta \log \theta + (1-\theta)\log(1-\theta) + \log 2.$$

The value of (9) for ABC is

$$k\{4(0) + 2[(\tfrac{1}{2}-h)\log(\tfrac{1}{2}-h) + (\tfrac{1}{2}+h)\log(\tfrac{1}{2}+h)] - 2[\tfrac{1}{2}\log\tfrac{1}{2} + \tfrac{1}{2}\log\tfrac{1}{2}]\}$$
$$= 2k[(\tfrac{1}{2}-h)\log(\tfrac{1}{2}-h) + (\tfrac{1}{2}+h)\log(\tfrac{1}{2}+h) + \log 2].$$

For BAC the value of (9) is

$$k\{2(0) + 2[h\log h + (1-h)\log(1-h) + 0]$$
$$- 4[(1-\tfrac{1}{2}h)\log(1-\tfrac{1}{2}h) + \tfrac{1}{2}h\log\tfrac{1}{2}h]\}.$$

The difference is

$$2k\{(\tfrac{1}{2}-h)\log(\tfrac{1}{2}-h) + (\tfrac{1}{2}+h)\log(\tfrac{1}{2}+h) + \log 2 - h\log h$$
$$- (1-h)\log(1-h) + 2[(1-\tfrac{1}{2}h)\log(1-\tfrac{1}{2}h) + \tfrac{1}{2}h\log\tfrac{1}{2}h]\}$$
$$= 2k\{(\tfrac{1}{2}-h)\log(1-2h) + (\tfrac{1}{2}+h)\log(1+2h) - h\log 2$$
$$- (1-h)\log(1-h) + (2-h)\log(1-\tfrac{1}{2}h)\}.$$

The limit as $h\to 0$ of this difference divided by h is $-\log 2<0$, hence for sufficiently small positive h, ABC is the ML ranking.

If the $\{S^{ij\nu}\}$ are all normally distributed with variance σ^2, then for ABC the value of (9) is

$$\frac{k}{2\sigma^2}\{2h^2\} = \frac{kh^2}{\sigma^2},$$

while for BAC the value of (9) is

$$\frac{k}{2\sigma^2}\{4(\tfrac{1}{2}h)^2\} = \frac{kh^2}{2\sigma^2},$$

so that BAC is the ML ranking.

THEOREM 8. *If $k_{ij}=k\geq 1$ for $i\neq j$, $i, j=1, 2, \cdots, m$, if $s_{a\nu}\geq s_{b\nu}$, $\nu=1, 2, \cdots, m$, and if all scores have distributions belonging to the same exponential family, then I_a precedes I_b in the ML ranking under the basic no-interaction model.* (Cf. Section 4.)

PROOF. We consider a ranking in which $I_a=I^i$, $I_b=I^j$, $i<j$, and prove that interchanging I_a and I_b can not decrease (8). Let e^{ij} denote the MLE's *after* interchange. The minimum value of (8) for the original ranking is therefore not greater than

$$k\sum_{r\neq t} T(s^{rt}) + \Theta[\tau(\hat{e}^{rt})] - s^{rt}\tau(\hat{e}^{rt}).$$

Also, $\hat{e}^{ir}\geq\hat{e}^{jr}$, $\hat{e}^{ri}\leq\hat{e}^{rj}$, $j=1, 2, \cdots, m$. The terms of this sum affected by the interchange are, before interchange,

$$
\begin{aligned}
&k \sum_{r \neq i,j} T(s_{ar}) + \Theta[\tau(\hat{e}^{ir})] - s_{ar}\tau(\hat{e}^{ir}) \\
&+ k \sum_{r \neq i,j} T(s_{br}) + \Theta[\tau(\hat{e}^{jr})] - s_{br}\tau(\hat{e}^{jr}) \\
&+ k \sum_{r \neq i,j} T(s_{ar}) + \Theta[\tau(\hat{e}^{ri})] - s_{ra}\tau(\hat{e}^{ri}) \\
&+ k \sum_{r \neq i,j} T(s_{rb}) + \Theta[\tau(\hat{e}^{rj})] - s_{rb}\tau(\hat{e}^{rj}) \\
&+ k\{T(s_{ab}) + \Theta[\tau(\hat{e}^{ij})] - s_{ab}\tau(\hat{e}^{ij})\} \\
&+ k\{T(s_{ba}) + \Theta[\tau(\hat{e}^{ji})] - s_{ba}\tau(\hat{e}^{ji})\}.
\end{aligned}
$$

After interchange they are the same, except that $\hat{e}^{ir}$ and $\hat{e}^{jr}$ are to be interchanged, as are $\hat{e}^{ri}$ and $\hat{e}^{rj}$, as well as $\hat{e}^{ij}$ and $\hat{e}^{ji}$. The difference is

$$
-k \sum_{r \neq i,j} (s_{ar} - s_{br})[\tau(\hat{e}^{ir}) - \tau(\hat{e}^{jr})] + k(s_{ab} - s_{ba})[\tau(\hat{e}^{ij}) - \tau(\hat{e}^{ji})] \leq 0,
$$

since τ is strictly increasing. Therefore the minimum value of (8) for the original ranking is not greater than the minimum value of (8) after interchange.

The scores matrix is *reducible* if there are a positive integer r less than m and a ranking such that in the corresponding scores matrix an $r \times (m-r)$ matrix of 0's stands in the lower left corner and an $(m-r) \times r$ matrix of 1's in the upper right corner.

COROLLARY. *Given a reducible observed scores matrix, the ML ranking of items corresponding to the rows distinguished by* 0's *is below that of others, if the distributions of the scores all belong to the same exponential family, and if* $k_{ij} = k \geq 1$ *for* $i \neq j$, $i, j = 1, 2, \cdots, m$.

PROOF. If I_b is an item corresponding to a row distinguished by 0's and I_a an item corresponding to a row not so distinguished, then either $s_{bv} = 0 \leq s_{av}$ or $s_{bv} \leq 1 = s_{av}$.

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TABLES OF CONFIDENCE LIMITS FOR THE BINOMIAL DISTRIBUTION

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Values of p (times 10,000) satisfying the equation

$$\alpha = \sum_{r=0}^{x} \binom{n}{r} p^r(1-p)^{n-r}$$

are given to four figures for $n=55(5)100$, $x=0(1)n-1$ and $\alpha=.005$, .010, .025, .050.

1. INTRODUCTION

GIVEN that there were x successes in n independent binomial trials, each having fixed probability of success, p, find $(1-2\alpha)$ 100% confidence limits for p of the form $p_1<p<p_2$. Neyman [7] has shown that since x is discrete the probability of being correct in stating $p_1<p<p_2$ is at least equal to $(1-2\alpha)$. If we desire confidence limits of the "central" type, then one would find p_1 by solving equation (1) and p_2 by solving equation (2), where

$$\alpha = \sum_{r=x}^{n} \binom{n}{r} p^r(1-p)^{n-r} \tag{1}$$

$$\alpha = \sum_{r=0}^{x} \binom{n}{r} p^r(1-p)^{n-r}. \tag{2}$$

It is only necessary to solve one of these equations, (2) say, for if we replace x by $n-x$ and p by $1-p$ in (2) the solution will be p_1. "Shorter" two-sided confidence limits for p have been computed by Crow [2], which, however, do not give efficient one-sided limits.

Table I gives to four significant figures the values of p (times 10,000) satisfying (2) for $n=55(5)100$, $x=0(1)n-1$, and $\alpha=.005$, .010, .025, .050. If $x=n$ the value of p is always unity and is therefore not tabled.

If the percentage points of the beta distribution are available, then it is well-known that p_1 and p_2 may be found from

$$\alpha = I_{p_1}(x, n-x+1) \tag{3}$$

$$\alpha = I_{1-p_2}(n-x, x+1) \tag{4}$$

where

$$I_p(a,b) = \int_0^p t^{a-1}(1-t)^{b-1}dt/B(a,b).$$

The values of p satisfying (2) have been found using the asymptotic series in powers of $(n-x/2)^{-2}$ given by equation (5.4) in Wise [9] if $n>4x+3$, and using equation (7.1) in Wise [9], but due to Carter, if $n\leq 4x+3$. Of the 20 values in Table I that can be checked using (4) without interpolation in Table 16 of [8], the maximum error found was three units in the fourth significant

figure. This occurred at $n=65$, $x=59$, $\alpha=.05$ where use of (4) gave $p_2=.9590$ while use of (7.1) gave $p_2=.9593$. This exact check is possible for five pairs of (n, x), namely (60, 0), (65, 5), (70, 59), (65, 59), (60, 59).

2. SOME USES FOR THE TABLE

A prime contractor receives shipments of parts from many different vendors. The number of defectives, x, in a random sample of size n is noted and confidence limits for, p, the per cent defective in the population are determined assuming that x follows a binomial distribution. Usually one-sided upper limits are desired of the type $p<p_2$, and if p_2 turns out to be greater than some predetermined value, the vendor involved may either be put on a "probation" list or dropped entirely. It is important in vendor rating and quality control problems to have tables of p_2 to facilitate the making of decisions. The tabulated values are of use in many other situations such as those mentioned in references [1], [3], [5].

Use of the table in finding p_1 does not guarantee four significant figures in all cases. Suppose, for example, that $n=55$, $x=5$ and $\alpha=.05$. We find from the table that $p_2=.1817$ and replacing x by $n-x$ we find that $p_1=1-.9638=.0362$. Here p_1 is found to only three significant figures. The probability of being correct in stating $.0362<p<.1817$ is then at least .90.

3. INTERPOLATION IN THE TABLE

In quality control work samples of five are quite common so that as data is accumulated the total sample size at any stage would be a multiple of five. This was the main reason for using a step size of five in n, in which case no interpolation would be needed. If interpolation with respect to n is desired, it turns out that linear interpolation in p is surprisingly good, yielding three significant figures in several cases tested. Linear interpolation in $\log p$ with respect to n was even better, yielding a maximum error of three in the fourth significant figure; however, this is somewhat inconvenient requiring use of logarithm and exponential tables. Use of five-point Lagrangian interpolation coefficients as tabulated in [6] yielded full accuracy of interpolates in several cases tested and is the recommended interpolation scheme.

4. RELATED TABLES

Mainland [5] lists to either two or three figures values of p satisfying equations (1) and (2) for $\alpha=.10, .025, .005$ and for the following values of x and n: Tables I*A* and I*B* $x=0(1)20$ and selected values of n up to 1000 in varying step sizes; Table II the percentage of x's in the sample $=.1(.2).7, 1(1)3, 5, 7.5, 10(5)50$ and selected values of n up to 100,000 in varying step sizes.

Grubbs [3] gives to three significant figures the values of p satisfying $.95=I_{1-p}(n-c, c+1)$ (his Table I), and $.10=I_{1-p}(n-c, c+1)$ (his Table II) for $n=1(1)150$ and $c=0(1)9$. His Table II represents an extension of our Table I. It is seen from his Table I that there is an overlap in our tables when $\alpha=.05$ for $x=n-10, n-9, \cdots, n-1$. On comparing the points of overlap, one must remember to replace his c with $n-x-1$ and to subtract his tabled entry from unity. Grubbs [3, p. 246] states " . . . tables . . . for which

$I_{1-p}(n-c, c+1) = .99, .95, .90, .10, .05, .01$ etc. would be of considerable practical use." Our Table I fulfills in part at least this requirement.

Clark [1] tables to four significant figures the values of p satisfying (1) for $n = 10(1)50$, $x = 1(1)n$ and $\alpha = .005, .010, .025, .050$.

Hald [4] gives to three decimals 95% and 99% two-sided confidence limits for p for selected combinations of n and x.

Pearson and Hartley [8] give in their Table 16, to five significant figures, values of p for which $\alpha = I_p(\nu_2/2, \nu_1/2)$ for $\alpha = .50, .25, .10, .05, .025, .01, .005$ and for selected combinations of ν_1 and ν_2. We can solve (1) for p_1 by putting $\nu_1 = 2(n-x+1)$, $\nu_2 = 2x$. We can solve (2) for p_2 by putting $\nu_1 = 2(x+1)$, $\nu_2 = 2(n-x)$ and subtracting the tabled entry from unity.

The equation $\alpha = I_p(a, b)$ can be solved graphically for either α or p to approximately two significant figures in the region $b \leq 60$, $a \leq 200$ using the charts of Table 17 in [8] when $a \geq b$. If $a < b$ we use the relation $I_p(a, b) = 1 - I_{1-p}(b, a)$.

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(Table follows on pp. 524–33)

TABLE I. VALUES OF p (TIMES 10,000) SUCH THAT

$$\alpha = \sum_{r=0}^{x} \binom{n}{r} p^r (1 - p)^{n-r}$$

$n = 55$

x \ α	.005	.010	.025	.050
0	918.4	803.2	648.7	530.1
1	1,275	1,147	971.9	833.7
2	1,579	1,442	1,253	1,101
3	1,856	1,713	1,512	1,350
4	2,117	1,968	1,759	1,587
5	2,366	2,213	1,995	1,817
6	2,605	2,448	2,225	2,040
7	2,837	2,677	2,448	2,258
8	3,062	2,900	2,666	2,472
9	3,282	3,117	2,880	2,682
10	3,497	3,331	3,090	2,889
11	3,708	3,540	3,297	3,093
12	3,915	3,746	3,501	3,294
13	4,118	3,948	3,702	3,493
14	4,318	4,148	3,900	3,690
15	4,514	4,344	4,096	3,885
16	4,708	4,538	4,290	4,077
17	4,899	4,730	4,481	4,268
18	5,088	4,919	4,671	4,458
19	5,274	5,106	4,858	4,645
20	5,457	5,290	5,044	4,831
21	5,639	5,473	5,227	5,015
22	5,817	5,653	5,409	5,198
23	5,994	5,831	5,589	5,380
24	6,168	6,007	5,768	5,559
25	6,341	6,182	5,945	5,738
26	6,511	6,354	6,120	5,915
27	6,678	6,524	6,293	6,091

x \ α	.005	.010	.025	.050
28	6,844	6,692	6,465	6,265
29	7,008	6,859	6,635	6,438
30	7,169	7,023	6,803	6,609
31	7,329	7,186	6,970	6,779
32	7,486	7,346	7,135	6,948
33	7,640	7,504	7,298	7,115
34	7,793	7,661	7,460	7,280
35	7,943	7,815	7,619	7,444
36	8,091	7,967	7,777	7,606
37	8,236	8,116	7,933	7,767
38	8,379	8,264	8,086	7,926
39	8,519	8,408	8,238	8,083
40	8,656	8,551	8,387	8,238
41	8,790	8,690	8,534	8,392
42	8,920	8,826	8,678	8,543
43	9,048	8,959	8,820	8,691
44	9,171	9,089	8,959	8,837
45	9,290	9,215	9,094	8,981
46	9,405	9,336	9,225	9,121
47	9,514	9,453	9,353	9,257
48	9,617	9,563	9,475	9,389
49	9,713	9,668	9,592	9,517
50	9,801	9,764	9,701	9,638
51	9,878	9,850	9,801	9,751
52	9,940	9,923	9,889	9,853
53	9,983	9,976	9,959	9,939
54	10,000	9,999	9,997	9,993

TABLE I—(*cont.*)

$n = 60$

x \ α	.005	.010	.025	.050
0	845.2	738.8	596.3	487.0
1	1,174	1,056	894.0	766.4
2	1,455	1,328	1,153	1,012
3	1,712	1,579	1,392	1,242
4	1,953	1,815	1,620	1,461
5	2,184	2,041	1,839	1,673
6	2,406	2,259	2,051	1,879
7	2,621	2,471	2,257	2,080
8	2,831	2,678	2,459	2,277
9	3,035	2,880	2,657	2,472
10	3,235	3,078	2,852	2,663
11	3,431	3,273	3,044	2,852
12	3,624	3,464	3,233	3,038
13	3,814	3,653	3,420	3,222
14	4,000	3,839	3,604	3,405
15	4,184	4,022	3,786	3,586
16	4,366	4,203	3,966	3,765
17	4,545	4,382	4,144	3,942
18	4,721	4,559	4,321	4,118
19	4,896	4,734	4,496	4,292
20	5,068	4,906	4,669	4,465
21	5,238	5,077	4,840	4,636
22	5,407	5,247	5,010	4,807
23	5,573	5,414	5,179	4,976
24	5,738	5,580	5,346	5,144
25	5,901	5,744	5,512	5,310
26	6,062	5,907	5,676	5,476
27	6,221	6,067	5,839	5,640
28	6,379	6,227	6,000	5,803
29	6,534	6,384	6,161	5,965

x \ α	.005	.010	.025	.050
30	6,688	6,541	6,319	6,126
31	6,841	6,695	6,477	6,286
32	6,991	6,848	6,633	6,444
33	7,140	6,999	6,788	6,602
34	7,287	7,149	6,941	6,758
35	7,432	7,297	7,093	6,913
36	7,576	7,444	7,244	7,067
37	7,717	7,589	7,393	7,220
38	7,857	7,732	7,541	7,371
39	7,995	7,873	7,687	7,521
40	8,130	8,012	7,832	7,670
41	8,264	8,150	7,975	7,818
42	8,395	8,285	8,116	7,964
43	8,525	8,419	8,256	8,108
44	8,651	8,550	8,393	8,251
45	8,776	8,679	8,529	8,392
46	8,897	8,806	8,663	8,532
47	9,016	8,930	8,794	8,669
48	9,132	9,051	8,923	8,805
49	9,244	9,169	9,049	8,938
50	9,352	9,283	9,172	9,068
51	9,457	9,394	9,292	9,196
52	9,556	9,500	9,408	9,321
53	9,650	9,601	9,520	9,442
54	9,738	9,696	9,626	9,558
55	9,818	9,784	9,726	9,668
56	9,888	9,863	9,818	9,772
57	9,945	9,929	9,899	9,865
58	9,985	9,978	9,963	9,944
59	10,000	9,999	9,997	9,993

TABLE I—(*cont.*)

$n = 65$

x \ α	.005	.010	.025	.050
0	782.8	684.0	551.7	450.4
1	1,088	978.1	827.6	709.1
2	1,349	1,231	1,068	937.1
3	1,588	1,464	1,290	1,150
4	1,813	1,683	1,501	1,353
5	2,028	1,894	1,705	1,550
6	2,235	2,097	1,902	1,741
7	2,436	2,295	2,094	1,928
8	2,631	2,487	2,282	2,111
9	2,822	2,676	2,466	2,292
10	3,009	2,861	2,648	2,470
11	3,193	3,043	2,827	2,645
12	3,373	3,222	3,003	2,819
13	3,551	3,398	3,177	2,991
14	3,726	3,572	3,349	3,160
15	3,898	3,744	3,519	3,329
16	4,069	3,913	3,687	3,496
17	4,237	4,081	3,854	3,661
18	4,403	4,247	4,019	3,825
19	4,567	4,411	4,183	3,988
20	4,729	4,573	4,345	4,149
21	4,890	4,734	4,506	4,310
22	5,049	4,893	4,665	4,469
23	5,206	5,051	4,823	4,627
24	5,361	5,207	4,980	4,784
25	5,515	5,362	5,136	4,940
26	5,668	5,516	5,290	5,095
27	5,819	5,668	5,444	5,250
28	5,968	5,818	5,596	5,403
29	6,116	5,968	5,747	5,555
30	6,263	6,116	5,897	5,706
31	6,408	6,263	6,046	5,856
32	6,552	6,408	6,193	6,006

x \ α	.005	.010	.025	.050
33	6,694	6,552	6,340	6,154
34	6,835	6,695	6,486	6,302
35	6,975	6,837	6,630	6,448
36	7,113	6,977	6,773	6,594
37	7,249	7,116	6,915	6,739
38	7,384	7,254	7,056	6,882
39	7,518	7,390	7,196	7,025
40	7,650	7,525	7,335	7,167
41	7,780	7,658	7,472	7,307
42	7,909	7,790	7,609	7,447
43	8,037	7,921	7,744	7,586
44	8,162	8,049	7,877	7,723
45	8,286	8,177	8,010	7,859
46	8,408	8,302	8,140	7,994
47	8,528	8,426	8,270	8,128
48	8,646	8,549	8,398	8,261
49	8,762	8,669	8,524	8,392
50	8,876	8,787	8,648	8,521
51	8,987	8,903	8,771	8,649
52	9,096	9,016	8,891	8,776
53	9,202	9,127	9,009	8,900
54	9,305	9,235	9,125	9,022
55	9,404	9,340	9,238	9,142
56	9,500	9,442	9,348	9,260
57	9,592	9,540	9,455	9,374
58	9,678	9,633	9,558	9,486
59	9,759	9,720	9,656	9,593
60	9,832	9,801	9,748	9,694
61	9,897	9,874	9,832	9,790
62	9,950	9,935	9,907	9,876
63	9,986	9,979	9,965	9,948
64	10,000	9,999	9,998	9,994

TABLE I—(*cont.*)

$n = 70$

x \ α	.005	.010	.025	.050
0	729.0	636.7	513.3	418.9
1	1,014	911.0	770.4	659.8
2	1,258	1,147	994.3	872.2
3	1,481	1,364	1,202	1,071
4	1,692	1,570	1,399	1,260
5	1,893	1,767	1,589	1,443
6	2,087	1,957	1,773	1,622
7	2,275	2,142	1,952	1,796
8	2,458	2,322	2,128	1,968
9	2,637	2,499	2,301	2,136
10	2,813	2,672	2,471	2,303
11	2,985	2,843	2,638	2,467
12	3,155	3,011	2,803	2,629
13	3,322	3,176	2,966	2,790
14	3,486	3,340	3,127	2,949
15	3,649	3,501	3,287	3,106
16	3,809	3,660	3,445	3,262
17	3,967	3,818	3,601	3,418
18	4,124	3,974	3,756	3,571
19	4,279	4,129	3,910	3,724
20	4,432	4,282	4,062	3,875
21	4,584	4,433	4,213	4,026
22	4,734	4,583	4,363	4,175
23	4,882	4,732	4,512	4,324
24	5,030	4,880	4,660	4,471
25	5,176	5,026	4,807	4,618
26	5,320	5,171	4,952	4,764
27	5,463	5,315	5,097	4,909
28	5,605	5,458	5,241	5,053
29	5,746	5,600	5,383	5,196
30	5,886	5,740	5,525	5,339
31	6,024	5,880	5,666	5,480
32	6,161	6,018	5,806	5,621
33	6,297	6,155	5,945	5,761
34	6,432	6,292	6,083	5,901

x \ α	.005	.010	.025	.050
35	6,565	6,427	6,220	6,039
36	6,697	6,561	6,356	6,177
37	6,829	6,693	6,491	6,314
38	6,959	6,825	6,626	6,450
39	7,087	6,956	6,759	6,586
40	7,215	7,086	6,892	6,721
41	7,341	7,214	7,023	6,854
42	7,466	7,341	7,154	6,987
43	7,590	7,468	7,283	7,120
44	7,712	7,593	7,412	7,251
45	7,833	7,716	7,539	7,381
46	7,953	7,839	7,666	7,511
47	8,071	7,960	7,791	7,640
48	8,188	8,080	7,915	7,767
49	8,304	8,199	8,038	7,894
50	8,418	8,316	8,160	8,020
51	8,530	8,432	8,281	8,145
52	8,641	8,546	8,400	8,268
53	8,749	8,658	8,518	8,390
54	8,856	8,769	8,634	8,511
55	8,961	8,878	8,749	8,631
56	9,064	8,985	8,862	8,749
57	9,164	9,090	8,974	8,866
58	9,262	9,192	9,083	8,981
59	9,357	9,292	9,190	9,094
60	9,449	9,389	9,295	9,205
61	9,537	9,483	9,396	9,314
62	9,622	9,574	9,495	9,420
63	9,702	9,660	9,590	9,523
64	9,776	9,741	9,681	9,622
65	9,844	9,816	9,766	9,717
66	9,904	9,883	9,845	9,805
67	9,953	9,939	9,913	9,885
68	9,987	9,981	9,968	9,952
69	10,000	9,999	9,998	9,994

TABLE I—(*cont.*)

$n = 75$

x \ α	.005	.010	.025	.050
0	682.1	595.6	480.0	391.6
1	949.3	852.6	720.6	616.9
2	1,178	1,074	930.3	815.7
3	1,388	1,278	1,125	1,001
4	1,585	1,471	1,310	1,179
5	1,774	1,655	1,488	1,351
6	1,957	1,834	1,660	1,518
7	2,134	2,008	1,829	1,682
8	2,306	2,177	1,994	1,842
9	2,475	2,343	2,156	2,001
10	2,640	2,507	2,316	2,157
11	2,803	2,667	2,473	2,311
12	2,963	2,825	2,628	2,463
13	3,120	2,981	2,781	2,614
14	3,275	3,135	2,933	2,763
15	3,429	3,287	3,083	2,911
16	3,580	3,438	3,232	3,058
17	3,730	3,587	3,379	3,204
18	3,878	3,734	3,525	3,348
19	4,024	3,880	3,670	3,492
20	4,169	4,024	3,814	3,635
21	4,313	4,168	3,956	3,776
22	4,455	4,310	4,098	3,917
23	4,596	4,451	4,238	4,057
24	4,736	4,590	4,378	4,196
25	4,874	4,729	4,517	4,334
26	5,011	4,867	4,654	4,472
27	5,147	5,003	4,791	4,609
28	5,283	5,139	4,927	4,745
29	5,416	5,273	5,062	4,880
30	5,549	5,407	5,196	5,015
31	5,681	5,539	5,330	5,149
32	5,812	5,671	5,462	5,282
33	5,942	5,802	5,594	5,414
34	6,070	5,931	5,725	5,546
35	6,198	6,060	5,855	5,678
36	6,325	6,188	5,985	5,808
37	6,450	6,315	6,114	5,938

x \ α	.005	.010	.025	.050
38	6,575	6,441	6,242	6,067
39	6,699	6,566	6,369	6,196
40	6,821	6,691	6,495	6,324
41	6,943	6,814	6,621	6,451
42	7,064	6,936	6,745	6,578
43	7,183	7,058	6,869	6,704
44	7,302	7,178	6,993	6,829
45	7,419	7,298	7,115	6,954
46	7,536	7,416	7,237	7,078
47	7,651	7,534	7,357	7,201
48	7,765	7,651	7,477	7,323
49	7,878	7,766	7,596	7,445
50	7,990	7,880	7,714	7,566
51	8,101	7,994	7,831	7,686
52	8,210	8,106	7,947	7,805
53	8,318	8,217	8,062	7,924
54	8,425	8,327	8,177	8,041
55	8,531	8,435	8,290	8,158
56	8,634	8,543	8,401	8,274
57	8,737	8,649	8,512	8,388
58	8,838	8,753	8,621	8,502
59	8,937	8,856	8,729	8,614
60	9,034	8,957	8,836	8,726
61	9,129	9,056	8,941	8,836
62	9,222	9,153	9,045	8,944
63	9,313	9,248	9,146	9,051
64	9,401	9,341	9,246	9,156
65	9,487	9,432	9,343	9,260
66	9,569	9,519	9,438	9,361
67	9,648	9,603	9,530	9,460
68	9,722	9,683	9,618	9,555
69	9,792	9,758	9,703	9,648
70	9,855	9,828	9,782	9,736
71	9,911	9,891	9,855	9,818
72	9,956	9,943	9,919	9,893
73	9,988	9,982	9,970	9,955
74	10,000	9,999	9,998	9,995

TABLE I—(*cont.*)

$n = 80$

x \ α	.005	.010	.025	.050
0	640.8	559.4	450.6	367.5
1	892.3	801.1	676.9	579.3
2	1,108	1,010	874.1	766.1
3	1,305	1,201	1,057	940.7
4	1,492	1,383	1,231	1,108
5	1,670	1,557	1,399	1,269
6	1,842	1,726	1,561	1,427
7	2,009	1,889	1,720	1,581
8	2,172	2,049	1,876	1,732
9	2,331	2,206	2,028	1,881
10	2,487	2,360	2,179	2,028
11	2,641	2,512	2,327	2,173
12	2,792	2,661	2,474	2,317
13	2,941	2,809	2,618	2,459
14	3,088	2,954	2,762	2,600
15	3,233	3,098	2,903	2,739
16	3,377	3,241	3,044	2,878
17	3,519	3,381	3,183	3,015
18	3,659	3,521	3,321	3,152
19	3,798	3,659	3,458	3,287
20	3,935	3,796	3,594	3,422
21	4,071	3,932	3,729	3,556
22	4,207	4,067	3,863	3,689
23	4,340	4,200	3,996	3,821
24	4,473	4,333	4,128	3,952
25	4,605	4,464	4,259	4,083
26	4,736	4,595	4,389	4,213
27	4,865	4,725	4,519	4,343
28	4,994	4,854	4,648	4,471
29	5,121	4,982	4,776	4,600
30	5,248	5,109	4,904	4,727
31	5,374	5,235	5,030	4,854
32	5,499	5,360	5,156	4,980
33	5,622	5,485	5,282	5,106
34	5,745	5,609	5,406	5,231
35	5,868	5,731	5,530	5,356
36	5,989	5,854	5,653	5,480
37	6,109	5,975	5,776	5,603
38	6,229	6,096	5,898	5,726
39	6,347	6,215	6,019	5,848

x \ α	.005	.010	.025	.050
40	6,465	6,334	6,140	5,970
41	6,582	6,453	6,259	6,091
42	6,698	6,570	6,379	6,212
43	6,813	6,687	6,497	6,332
44	6,928	6,803	6,615	6,451
45	7,041	6,918	6,732	6,570
46	7,154	7,032	6,849	6,688
47	7,266	7,145	6,965	6,806
48	7,376	7,258	7,080	6,923
49	7,486	7,370	7,195	7,040
50	7,595	7,481	7,308	7,156
51	7,703	7,591	7,421	7,271
52	7,810	7,700	7,534	7,386
53	7,916	7,809	7,645	7,500
54	8,021	7,916	7,756	7,613
55	8,125	8,022	7,866	7,726
56	8,228	8,128	7,975	7,838
57	8,330	8,232	8,083	7,949
58	8,431	8,336	8,190	8,060
59	8,530	8,438	8,296	8,169
60	8,628	8,539	8,402	8,278
61	8,725	8,639	8,506	8,386
62	8,820	8,738	8,609	8,493
63	8,914	8,835	8,711	8,599
64	9,007	8,931	8,812	8,704
65	9,097	9,025	8,912	8,808
66	9,186	9,118	9,010	8,911
67	9,273	9,208	9,106	9,012
68	9,358	9,297	9,201	9,112
69	9,440	9,384	9,294	9,211
70	9,520	9,468	9,385	9,307
71	9,597	9,550	9,474	9,402
72	9,670	9,628	9,560	9,494
73	9,740	9,703	9,643	9,584
74	9,805	9,774	9,722	9,670
75	9,864	9,839	9,796	9,753
76	9,916	9,898	9,864	9,830
77	9,959	9,947	9,924	9,899
78	9,989	9,983	9,972	9,958
79	10,000	9,999	9,998	9,995

TABLE I—(*cont.*)

$n = 85$

x \ α	.005	.010	.025	.050
0	604.3	527.4	424.7	346.3
1	841.8	755.6	638.1	546.0
2	1,045	952.4	824.2	722.2
3	1,232	1,134	997.0	887.0
4	1,408	1,305	1,161	1,045
5	1,577	1,470	1,320	1,197
6	1,740	1,629	1,473	1,346
7	1,898	1,784	1,623	1,491
8	2,052	1,936	1,771	1,634
9	2,203	2,084	1,915	1,775
10	2,351	2,230	2,057	1,914
11	2,497	2,374	2,198	2,051
12	2,640	2,515	2,336	2,187
13	2,782	2,655	2,473	2,321
14	2,921	2,793	2,609	2,454
15	3,059	2,929	2,743	2,586
16	3,195	3,065	2,876	2,718
17	3,330	3,198	3,008	2,848
18	3,463	3,331	3,139	2,977
19	3,595	3,462	3,269	3,105
20	3,726	3,592	3,397	3,233
21	3,856	3,721	3,525	3,360
22	3,984	3,849	3,652	3,486
23	4,112	3,976	3,779	3,611
24	4,238	4,102	3,904	3,735
25	4,363	4,227	4,029	3,859
26	4,488	4,352	4,153	3,983
27	4,611	4,475	4,276	4,105
28	4,734	4,598	4,398	4,228
29	4,856	4,720	4,520	4,349
30	4,977	4,841	4,641	4,470
31	5,097	4,961	4,762	4,591
32	5,216	5,081	4,882	4,711
33	5,335	5,200	5,001	4,830
34	5,452	5,318	5,120	4,949
35	5,569	5,435	5,238	5,067
36	5,685	5,552	5,355	5,185
37	5,801	5,668	5,472	5,303
38	5,915	5,784	5,589	5,420
39	6,029	5,898	5,704	5,536
40	6,142	6,012	5,819	5,652
41	6,255	6,126	5,934	5,767
42	6,366	6,238	6,048	5,882

x \ α	.005	.010	.025	.050
43	6,477	6,350	6,161	5,997
44	6,587	6,462	6,274	6,111
45	6,697	6,572	6,387	6,225
46	6,805	6,682	6,498	6,338
47	6,913	6,792	6,610	6,450
48	7,021	6,900	6,720	6,562
49	7,127	7,008	6,830	6,674
50	7,233	7,115	6,939	6,785
51	7,337	7,222	7,048	6,895
52	7,441	7,328	7,156	7,005
53	7,545	7,433	7,264	7,115
54	7,647	7,537	7,371	7,224
55	7,749	7,641	7,477	7,332
56	7,850	7,743	7,583	7,440
57	7,949	7,845	7,688	7,548
58	8,048	7,947	7,792	7,654
59	8,147	8,047	7,896	7,761
60	8,244	8,146	7,998	7,866
61	8,340	8,245	8,100	7,971
62	8,435	8,343	8,202	8,075
63	8,529	8,439	8,302	8,179
64	8,622	8,535	8,402	8,281
65	8,714	8,630	8,500	8,383
66	8,804	8,723	8,598	8,485
67	8,894	8,816	8,695	8,585
68	8,982	8,907	8,790	8,684
69	9,068	8,996	8,885	8,783
70	9,153	9,085	8,978	8,880
71	9,236	9,172	9,070	8,977
72	9,318	9,257	9,161	9,072
73	9,397	9,340	9,250	9,166
74	9,474	9,421	9,337	9,258
75	9,549	9,501	9,422	9,349
76	9,621	9,577	9,506	9,438
77	9,690	9,651	9,586	9,524
78	9,756	9,721	9,664	9,609
79	9,817	9,787	9,738	9,690
80	9,872	9,849	9,808	9,767
81	9,921	9,904	9,872	9,840
82	9,962	9,950	9,929	9,905
83	9,989	9,984	9,974	9,960
84	10,000	9,999	9,998	9,995

TABLE I—(*cont.*)

$n = 90$

x \ α	.005	.010	.025	.050
0	571.7	498.8	401.6	327.4
1	796.7	714.9	603.6	516.3
2	989.7	901.4	779.8	683.0
3	1,167	1,073	943.4	839.0
4	1,334	1,236	1,099	988.2
5	1,494	1,392	1,249	1,133
6	1,648	1,543	1,395	1,273
7	1,799	1,690	1,537	1,411
8	1,945	1,834	1,677	1,547
9	2,089	1,975	1,814	1,680
10	2,229	2,114	1,949	1,812
11	2,368	2,250	2,082	1,942
12	2,504	2,384	2,213	2,071
13	2,638	2,517	2,343	2,198
14	2,771	2,648	2,472	2,324
15	2,902	2,778	2,600	2,450
16	3,032	2,907	2,726	2,574
17	3,160	3,034	2,851	2,698
18	3,287	3,160	2,975	2,820
19	3,413	3,285	3,099	2,942
20	3,538	3,408	3,221	3,063
21	3,661	3,531	3,343	3,183
22	3,784	3,653	3,464	3,303
23	3,905	3,774	3,584	3,422
24	4,026	3,894	3,703	3,541
25	4,146	4,014	3,822	3,659
26	4,264	4,132	3,940	3,776
27	4,382	4,250	4,057	3,893
28	4,500	4,367	4,174	4,009
29	4,616	4,484	4,290	4,124
30	4,732	4,599	4,406	4,240
31	4,847	4,714	4,520	4,354
32	4,961	4,829	4,635	4,468
33	5,074	4,942	4,749	4,582
34	5,187	5,055	4,862	4,695
35	5,299	5,168	4,974	4,808
36	5,410	5,279	5,087	4,920
37	5,521	5,390	5,198	5,032
38	5,631	5,501	5,309	5,144
39	5,740	5,611	5,420	5,255
40	5,849	5,720	5,530	5,366
41	5,957	5,829	5,640	5,476
42	6,064	5,937	5,749	5,586
43	6,171	6,044	5,857	5,695
44	6,277	6,151	5,965	5,804

x \ α	.005	.010	.025	.050
45	6,382	6,258	6,073	5,912
46	6,487	6,364	6,180	6,020
47	6,591	6,469	6,287	6,128
48	6,694	6,573	6,393	6,235
49	6,797	6,677	6,499	6,342
50	6,899	6,781	6,604	6,449
51	7,001	6,884	6,708	6,555
52	7,102	6,986	6,812	6,660
53	7,202	7,088	6,916	6,765
54	7,301	7,189	7,019	6,870
55	7,400	7,289	7,122	6,974
56	7,498	7,389	7,224	7,078
57	7,596	7,488	7,325	7,181
58	7,693	7,586	7,426	7,284
59	7,789	7,684	7,526	7,386
60	7,884	7,781	7,626	7,488
61	7,978	7,878	7,725	7,590
62	8,072	7,973	7,824	7,691
63	8,165	8,068	7,922	7,791
64	8,257	8,162	8,019	7,891
65	8,348	8,256	8,115	7,990
66	8,438	8,348	8,211	8,088
67	8,527	8,440	8,306	8,186
68	8,616	8,531	8,401	8,284
69	8,703	8,621	8,494	8,381
70	8,789	8,710	8,587	8,477
71	8,874	8,798	8,679	8,572
72	8,958	8,884	8,770	8,666
73	9,041	8,970	8,860	8,760
74	9,122	9,055	8,949	8,853
75	9,202	9,138	9,037	8,944
76	9,280	9,219	9,124	9,035
77	9,357	9,300	9,209	9,125
78	9,432	9,378	9,293	9,213
79	9,505	9,455	9,375	9,300
80	9,575	9,529	9,455	9,386
81	9,643	9,601	9,534	9,470
82	9,708	9,671	9,610	9,551
83	9,770	9,737	9,683	9,631
84	9,827	9,799	9,753	9,707
85	9,880	9,857	9,819	9,780
86	9,926	9,909	9,880	9,849
87	9,964	9,953	9,933	9,911
88	9,990	9,985	9,975	9,963
89	10,000	10,000	9,998	9,995

TABLE I—(*cont.*)

$n = 95$

x \ α	.005	.010	.025	.050
0	542.4	473.2	380.9	310.4
1	756.2	678.4	572.6	489.6
2	939.6	855.5	739.9	647.9
3	1,108	1,019	895.2	796.0
4	1,267	1,173	1,043	937.6
5	1,419	1,322	1,186	1,075
6	1,566	1,466	1,324	1,208
7	1,709	1,606	1,459	1,339
8	1,849	1,742	1,592	1,468
9	1,985	1,877	1,722	1,595
10	2,119	2,008	1,851	1,720
11	2,251	2,138	1,977	1,844
12	2,381	2,266	2,103	1,966
13	2,509	2,393	2,226	2,087
14	2,636	2,518	2,349	2,207
15	2,761	2,642	2,470	2,327
16	2,884	2,764	2,590	2,445
17	3,007	2,885	2,710	2,562
18	3,128	3,005	2,828	2,679
19	3,248	3,124	2,946	2,795
20	3,367	3,243	3,062	2,910
21	3,485	3,360	3,178	3,025
22	3,602	3,476	3,294	3,139
23	3,718	3,592	3,408	3,252
24	3,834	3,707	3,522	3,365
25	3,948	3,821	3,635	3,477
26	4,062	3,934	3,748	3,589
27	4,175	4,047	3,860	3,700
28	4,287	4,158	3,971	3,811
29	4,398	4,270	4,082	3,921
30	4,509	4,380	4,192	4,031
31	4,619	4,490	4,302	4,141
32	4,728	4,600	4,411	4,250
33	4,837	4,708	4,520	4,358
34	4,945	4,817	4,628	4,466
35	5,053	4,924	4,736	4,574
36	5,159	5,031	4,843	4,681
37	5,266	5,138	4,950	4,788
38	5,371	5,244	5,056	4,894
39	5,476	5,349	5,162	5,000
40	5,581	5,454	5,267	5,106
41	5,684	5,558	5,372	5,211
42	5,788	5,662	5,477	5,316
43	5,890	5,766	5,581	5,421
44	5,992	5,868	5,685	5,525
45	6,094	5,971	5,788	5,629
46	6,195	6,072	5,890	5,733
47	6,295	6,174	5,993	5,836

x \ α	.005	.010	.025	.050
48	6,395	6,274	6,095	5,938
49	6,495	6,375	6,196	6,041
50	6,593	6,474	6,297	6,143
51	6,691	6,574	6,398	6,244
52	6,789	6,672	6,498	6,346
53	6,886	6,770	6,598	6,447
54	6,982	6,868	6,697	6,547
55	7,078	6,965	6,796	6,647
56	7,174	7,062	6,894	6,747
57	7,268	7,158	6,992	6,846
58	7,362	7,253	7,090	6,945
59	7,456	7,348	7,187	7,044
60	7,549	7,443	7,283	7,142
61	7,641	7,536	7,379	7,240
62	7,733	7,630	7,475	7,337
63	7,823	7,722	7,570	7,434
64	7,914	7,814	7,664	7,531
65	8,003	7,906	7,758	7,627
66	8,092	7,997	7,852	7,723
67	8,180	8,087	7,944	7,818
68	8,268	8,176	8,037	7,912
69	8,354	8,265	8,128	8,006
70	8,440	8,353	8,219	8,100
71	8,525	8,440	8,310	8,193
72	8,609	8,527	8,400	8,286
73	8,693	8,612	8,489	8,378
74	8,775	8,697	8,577	8,469
75	8,856	8,781	8,665	8,560
76	8,937	8,864	8,751	8,650
77	9,016	8,946	8,837	8,739
78	9,094	9,027	8,922	8,827
79	9,170	9,106	9,006	8,915
80	9,246	9,185	9,089	9,002
81	9,320	9,262	9,171	9,087
82	9,392	9,338	9,252	9,172
83	9,463	9,412	9,331	9,256
84	9,532	9,484	9,409	9,338
85	9,598	9,555	9,485	9,419
86	9,662	9,623	9,559	9,498
87	9,724	9,688	9,631	9,575
88	9,782	9,751	9,700	9,651
89	9,836	9,810	9,766	9,723
90	9,886	9,865	9,829	9,792
91	9,930	9,914	9,886	9,857
92	9,966	9,955	9,936	9,915
93	9,990	9,986	9,976	9,965
94	10,000	10,000	9,998	9,996

TABLE I—(*cont.*)

$n = 100$

x \ α	.005	.010	.025	.050
0	516.0	450.1	362.2	295.1
1	719.6	645.4	544.6	465.6
2	894.3	814.1	703.8	616.2
3	1,055	969.7	851.8	757.1
4	1,206	1,117	992.6	892.0
5	1,351	1,259	1,128	1,023
6	1,492	1,396	1,260	1,150
7	1,628	1,529	1,389	1,275
8	1,761	1,659	1,516	1,397
9	1,892	1,787	1,640	1,518
10	2,020	1,913	1,762	1,637
11	2,145	2,037	1,883	1,755
12	2,270	2,160	2,002	1,872
13	2,392	2,280	2,120	1,987
14	2,513	2,400	2,237	2,102
15	2,632	2,518	2,353	2,215
16	2,751	2,635	2,468	2,328
17	2,868	2,750	2,582	2,440
18	2,984	2,865	2,695	2,551
19	3,098	2,979	2,807	2,662
20	3,212	3,092	2,918	2,772
21	3,325	3,204	3,029	2,881
22	3,437	3,316	3,139	2,990
23	3,549	3,426	3,249	3,098
24	3,659	3,536	3,357	3,206
25	3,768	3,645	3,466	3,313
26	3,877	3,753	3,573	3,420
27	3,986	3,861	3,680	3,526
28	4,093	3,968	3,787	3,632
29	4,200	4,075	3,893	3,737
30	4,306	4,181	3,998	3,842
31	4,412	4,286	4,103	3,947
32	4,517	4,391	4,208	4,051
33	4,621	4,495	4,312	4,155
34	4,725	4,599	4,415	4,258
35	4,828	4,702	4,518	4,361
36	4,930	4,805	4,621	4,463
37	5,032	4,907	4,724	4,566
38	5,134	5,009	4,825	4,668
39	5,235	5,110	4,927	4,769
40	5,335	5,211	5,028	4,870
41	5,435	5,311	5,129	4,971
42	5,535	5,411	5,229	5,072
43	5,633	5,510	5,329	5,172
44	5,732	5,609	5,428	5,272
45	5,830	5,708	5,527	5,371
46	5,927	5,806	5,626	5,470
47	6,024	5,903	5,724	5,569
48	6,120	6,000	5,822	5,668
49	6,216	6,097	5,920	5,766

x \ α	.005	.010	.025	.050
50	6,311	6,193	6,017	5,864
51	6,406	6,289	6,114	5,961
52	6,501	6,384	6,210	6,059
53	6,595	6,479	6,306	6,156
54	6,688	6,573	6,402	6,252
55	6,781	6,667	6,497	6,348
56	6,873	6,760	6,592	6,444
57	6,965	6,853	6,686	6,540
58	7,056	6,946	6,780	6,635
59	7,147	7,038	6,874	6,730
60	7,238	7,129	6,967	6,825
61	7,327	7,220	7,060	6,919
62	7,417	7,311	7,152	7,013
63	7,505	7,401	7,244	7,107
64	7,593	7,491	7,336	7,200
65	7,681	7,580	7,427	7,293
66	7,768	7,668	7,518	7,385
67	7,854	7,756	7,608	7,477
68	7,940	7,844	7,698	7,569
69	8,025	7,931	7,787	7,660
70	8,110	8,017	7,876	7,751
71	8,194	8,103	7,965	7,841
72	8,277	8,188	8,052	7,932
73	8,360	8,273	8,140	8,021
74	8,442	8,356	8,226	8,110
75	8,523	8,440	8,313	8,199
76	8,603	8,522	8,398	8,287
77	8,683	8,604	8,483	8,375
78	8,762	8,685	8,567	8,462
79	8,839	8,765	8,651	8,548
80	8,916	8,845	8,734	8,634
81	8,992	8,923	8,816	8,719
82	9,067	9,001	8,898	8,804
83	9,141	9,077	8,978	8,888
84	9,214	9,153	9,058	8,971
85	9,285	9,227	9,136	9,053
86	9,355	9,300	9,214	9,134
87	9,424	9,372	9,290	9,214
88	9,491	9,442	9,365	9,294
89	9,556	9,511	9,439	9,372
90	9,619	9,577	9,511	9,448
91	9,680	9,642	9,581	9,524
92	9,738	9,704	9,650	9,597
93	9,793	9,764	9,715	9,668
94	9,845	9,820	9,778	9,737
95	9,892	9,872	9,837	9,803
96	9,933	9,918	9,892	9,864
97	9,967	9,958	9,940	9,920
98	9,991	9,987	9,978	9,966
99	10,000	10,000	9,998	9,996

A METHOD OF ANALYZING LOG-NORMALLY DISTRIBUTED SURVIVAL DATA WITH INCOMPLETE FOLLOW-UP*

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A method is presented for analyzing survival data with incomplete follow-up when there is reason to suppose that the data have a translated log-normal distribution. The life table method is used to make allowances for those patients lost to observation and for tabulating the resulting survival frequencies. Modified maximum likelihood equations and short-cut procedures are presented for calculating the parameters of the general three parameter log-normal distribution from grouped data as represented by the life table.

1. INTRODUCTION

MANY variables encountered in practice have distributions which can be more or less closely approximated by theoretical frequency curves. When the approximation is known to be close, the theoretical frequency curve provides a compact way of summarizing the data in terms of a few parameters and provides a smooth curve for describing the observed distribution. Furthermore, parametric distributions lend themselves to methods of testing for differences between populations which are usually more powerful than corresponding non-parametric techniques and, in some instances, the mathematical distribution may help to elucidate the nature of the phenomenon under investigation.

Several investigators (e.g. Boag [1], Lea [7], Osgood [11, 12, 13], and Tivey [13, 14, 15]) have observed that the distributions of duration of survival in several diseases can be rather closely approximated by a log-normal distribution. They have pointed out that although the frequency distributions of survival times in these diseases are markedly skewed to the right, the logarithms of the survival times are approximately normally distributed. Thus, if one were to plot the cumulative distribution of the survival times of patients with these diseases on log-normal graph paper (Codex Book Co., Inc., No. 31376) a straight line should be obtained. An inspection of the graphs presented by these authors often showed, however, that the observed distribution was markedly convex with respect to the predicted straight line. A similar disparity was observed in a recent study [4] of survival of patients suffering from chronic leukemia (Fig. 1).

Gaddum [5] has shown that such deviations can be corrected by subtracting an appropriate constant from the survival times. This transformation results in a three-parameter or translated log-normal distribution. That is, the variable t defined by

$$t = \frac{1}{c} \log \left(\frac{x - a}{b} \right) \tag{1}$$

is a standard normal variate with mean zero and variance one, where x is the

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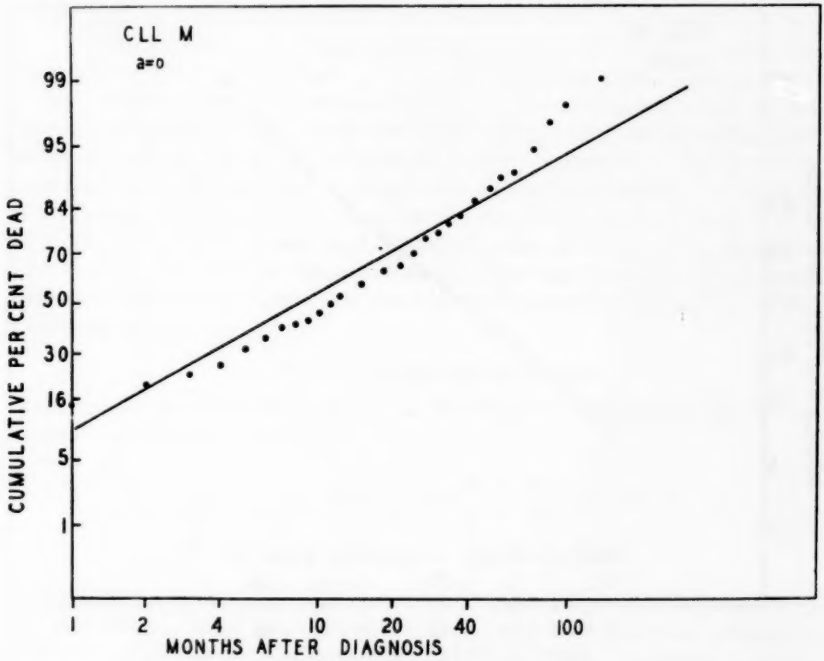

FIG. 1. Duration of survival from date of diagnosis of 234 patients with chronic lymphocytic leukemia. Duration of survival in months is plotted along the logarithmic scale. The straight line was calculated by Lea's method [7].

duration of survival measured from date of diagnosis, onset of symptoms, or some other point in the course of the disease, a is the appropriate constant, and log b and c are the mean and standard deviation of the variable log $(x-a)$, respectively.

To date no method has been proposed for fitting a translated log-normal distribution to incomplete data, although maximum likelihood solutions for complete data have been provided by Cohen [2]. Modifications of Cohen's equations applicable to life data, as employed in a recent study of survival in chronic leukemia [4], are presented below. It has not yet been ascertained whether the values obtained by this modified maximum likelihood procedure are in complete agreement with the exact maximum likelihood solution for the case of incomplete data. Since the latter procedure is not yet available, the method proposed below must rest on heuristic grounds. It may be noted, however, that if the survival data are complete, the procedure proposed below is strictly equivalent to the maximum likelihood solution for grouped data.

2. VALIDITY AND INTERPRETATION OF THE LOG-NORMAL DISTRIBUTION

Before proceeding to the method of estimating the parameters of the translated log-normal distribution some non-mathematical problems of interpreta-

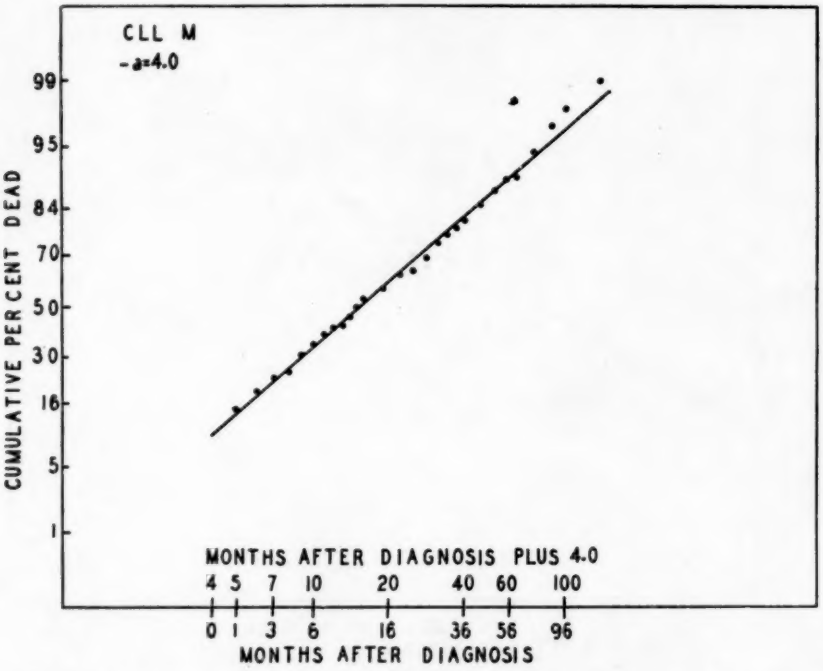

FIG. 2. Duration of survival from date of diagnosis of 234 male patients with chronic lymphocytic leukemia. Duration of survival in months plus 4.0 is plotted along the logarithmic scale. The straight line was calculated by the method described in the text.

tion and validity should first be considered. The first problem is the validity of assuming that survival data follow a translated log-normal distribution. This problem can be resolved only in terms of empirical observations. The authors noted above have shown that the logarithms of survival times are approximately normally distributed. Fig. 2 shows that the subtraction of an appropriate constant may improve the approximation. Hence, empirically it seems valid to assume that the translated log-normal distribution is appropriate.

The second consideration is whether the use of a third parameter, the constant to be subtracted, appreciably improves the goodness-of-fit. A comparison of Fig. 1 with Fig. 2 shows that the improvement is appreciable. In fact, an exceedingly poor fit (P is approximately .001) is changed to a good fit (P is approximately .66). Thus, it seems justifiable, in certain cases, to use the more general three-parameter distribution in preference to the two-parameter distribution (i.e. where a is assumed *a priori* to be zero).

The third problem is the interpretation to be given to the three parameters and the logarithmic transformation. Any venture at a biological interpretation of a mathematical phenomenon is hazardous. In general, however, a translated log-normal distribution might be interpreted as meaning that a multitude of factors affect the variable under consideration, that the effect of each factor de-

pends upon the accumulative effect of the factors which have preceded it, and that the factors apparently began to act at the point "a" in the original time scale. The point "a", however, is not necessarily the time of onset of the disease, nor need it be any other clinically recognizable point in the natural history of the disease. It is strictly speaking nothing more than a mathematical estimation of a parameter which enables us to fit a theoretical distribution to an empirical one. If one desires to interpret it biologically, it might be described as the apparent origin in time at which the cumulative factors which lead to death began to act. It should be emphasized again, that these interpretations may have no real basis in observable facts. They are, however, in agreement with some theoretical models that have been suggested for the etiology of neoplastic disorders [10].

3. MAXIMUM LIKELIHOOD EQUATIONS

Under the condition that $\log (x-a)$ is normally distributed, the density function of x would be given by

$$f(x) = \begin{cases} \dfrac{1}{c(x-a)\sqrt{2\pi}} \exp\left[\dfrac{-1}{2c^2}\log^2\left(\dfrac{x-a}{b}\right)\right] & \text{if} \quad x > a \\ 0 \quad \text{if} \quad x \leq a \end{cases} \tag{2}$$

If the duration of survival of each patient were known, the parameters of this distribution could be obtained from the maximum likelihood estimators described by Cohen [2]. He has found that the maximum likelihood estimates of $\log b$ and c^2 are given by

$$\log \hat{b} = \frac{1}{n}\sum_{i=1}^{n} \log (x_i - \hat{a}) \tag{3}$$

and

$$\hat{c}^2 = \frac{1}{n}\sum_{i=1}^{n} \log^2 (x_i - \hat{a}) - \left(\frac{1}{n}\sum_{i=1}^{n} \log (x_i - \hat{a})\right)^2 \tag{4}$$

while the maximum likelihood estimate of a is the solution, $\hat{a}$, of the equation

$$\lambda(a) = \left[\sum_{i=1}^{n} \frac{1}{x_i - a}\right]\left[n\sum_{i=1}^{n} \log (x_i - a) - n\sum_{i=1}^{n} \log^2 (x_i - a) + \left(\sum_{i=1}^{n} \log (x_i - a)\right)^2\right] - n^2 \sum_{i=1}^{n} \frac{\log (x_i - a)}{x_i - a} = 0 \tag{5}$$

where n is the number of patients observed and x_i is the duration of survival of the ith patient. $\lambda(a)=0$ may be solved for $\hat{a}$ by an iterative procedure and the solution subsequently substituted in the preceding equations to obtain the estimates $\log \hat{b}$ and $\hat{c}^2$.

4. ADJUSTMENT FOR INCOMPLETE FOLLOW-UP

In human survival studies, however, the duration of survival is generally not known for each individual. Some patients may still be alive at the termination

of the study, while others may withdraw from the study prior to their deaths and be lost to follow-up. For such patients the value of x_i would be unknown.

One method of making allowances for these patients is the actuarial or life table method. The life table method is based on the calculation of conditional probabilities of dying during short intervals of the period under consideration. A commonly used expression for calculating the probabilities of dying during any interval, is known as the "$\frac{1}{2}$-rule" and is given by the following equation:*

$$q_i = \frac{D_i}{L_i - \frac{1}{2}W_i} \tag{6}$$

Harris, Meier, and Tukey [6] have pointed out "that this '$\frac{1}{2}$-rule' is not an assumption, but the natural, mathematical, first approximation to maximum likelihood in a situation where even a second approximation is rarely needed."

Having calculated the values of q_i for each interval, these are then applied to a hypothetical number of individuals, l_1, alive at the beginning of the first interval, e.g. 1,000, to yield l_i, the number of individuals of the original 1,000 who are alive at the beginning of the ith interval, and d_i, the number of deaths that occur during the ith interval, for i ranging from 1 to p. The values of the l_i and the d_i may be obtained from the following relations:

$$d_i = q_i \cdot l_i \tag{7}$$

$$l_{i+1} = l_i - d_i \tag{8}$$

The adjusted survival curve of a group of patients may next be obtained by plotting the proportion who have died as a function of time. This is done by plotting the cumulative sum of d_i/l_1 in the life table against the endpoint of the last interval in each sum.

5. ESTIMATION OF THE PARAMETERS

An intuitive approach toward estimating the parameters of the log-normal distribution would be to treat the numbers dying in each interval as calculated in the life table, the d_i, as if they had arisen from an ordinary random sample. This was the method adopted in the study on chronic leukemia mentioned previously. Before substituting the values obtained from the life table into the usual maximum likelihood equations, however, certain adjustments must be made for the grouping.

When using grouped data, the assumption is usually made that all the values within any small interval can be considered to be concentrated at the mean

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* The usual life table notation has been slightly modified to allow for summation over the index for intervals rather than directly over time. The following notation is employed:

y_i—the beginning of the ith interval.

L_i—the number of individuals alive and under observation at the beginning of the ith interval, i.e. at time y_i.

D_i—the number of individuals known to have died during the ith interval.

W_i—the number of individuals withdrawing during the ith interval.

q_i—the conditional probability of dying during the ith interval, given survival to y_i.

If the number of intervals in the life table is p, the initial point of the time scale, y_1, is set equal to the (unknown) parameter α, y_2 is chosen to be positive and larger than α, and the upper endpoint of the last interval, y_{p+1}, is set equal to plus infinitity, while the remaining $p-2$ points are placed at any convenient distances between y_2 and y_{p+1}. Since the value of α is not known at the outset of the calculations, y_1 may be taken to be the origin from which survival is measured and set equal to zero.

of the portion of the distribution over that interval. That is, in terms of the transformed variable, t, it is desired to find the point t_i^* for each interval such that

$$t_i^* \int_{t_i}^{t_{i+1}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt = \int_{t_i}^{t_{i+1}} \frac{1}{\sqrt{2\pi}} te^{-t^2/2}\,dt \tag{9}$$

where

$$t_i = \frac{1}{c} \log \frac{(y_i - a)}{b} \tag{10}$$

For all intervals but the first and the last, the usual assumption may be made that the desired point is the midpoint of the transformed interval, i.e.

$$t_i^* = \tfrac{1}{2}(t_i + t_{i+1}) \qquad i = 2, 3, \cdots, p-1 \tag{11}$$

The point in each interval on the original time scale which is mapped into the t_i^* is given by the inverse relationship

$$t_i^* = \frac{1}{c} \log \left(\frac{y_i^* - a}{b} \right) \tag{12}$$

Substituting (12) in (11), the following simplification is obtained

$$\log (y_i^* - a) = \tfrac{1}{2}(\log (y_i - a) + \log (y_{i+1} - a)) \qquad i = 2, 3, \cdots, p-1 \tag{13}$$

For the first and last intervals Eqns. (11) and (13) do not hold since $y_1 = a$, $y_{p+1} = \infty$, $t_1 = -\infty$, and $t_{p+1} = \infty$. Estimates of the means of these intervals can be made from the data in the life table by making the assumptions that

$$\frac{d_1}{l_1} = \int_{-\infty}^{t_2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \tag{14}$$

and

$$\frac{d_p}{l_1} = \int_{t_p}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \tag{15}$$

Equation (9) can then be solved explicitly to yield

$$t_1^* = \frac{l_1}{d_1} \left[\frac{-e^{-t_2^2/2}}{\sqrt{2\pi}} \right] \tag{16}$$

and

$$t_p^* = \frac{l_1}{d_p} \left[\frac{e^{-t_p^2/2}}{\sqrt{2\pi}} \right] \tag{17}$$

where t_2 and t_p are determined from (14) and (15), respectively.

It is to be observed that the values of y_i^* $(i = 2, 3, \cdots, p-1)$ as determined by Eqn. (13) are independent of b and c, and depend only on a. The values of y_1^* and y_p^*, however, obtained by substituting the values of t_1^* and t_p^* found from Eqns. (16) and (17) into Eqn. (12) depend on all three parameters.

A method of determining y_1^* and y_p^* which is independent of the parameter

b is provided by the relations

$$\log (y_1^* - a) = \log (y_2 - a) - c(t_2 - t_1^*) \tag{18}$$

and

$$\log (y_p^* - a) = \log (y_p - a) + c(t_p^* - t_p) \tag{19}$$

where t_2, t_p, t_1^*, and t_p^* are determined by Equations (14) through (17), respectively. Initial estimates of the parameters a, b, and c may be made by short-cut methods to be discussed below.

The above relations provide convenient estimates for the y_i^* to be used in the following modifications of Cohen's maximum likelihood equations which are to be used in the estimation of the parameters of the log-normal distribution from the grouped data of the life table:

$$\log \hat{b} = \frac{1}{l_1} \sum_{i=1}^{p} d_i \log (y_i - \hat{a}) \tag{20}$$

$$\hat{c}^2 = \frac{1}{l_1} \sum_{i=1}^{p} d_i \log^2 (y_i^* - \hat{a}) - \left[\frac{1}{l_1} \sum_{i=1}^{p} d_i \log (y_i^* - \hat{a}) \right]^2 \tag{21}$$

$$\begin{aligned} \lambda(a) = \left[\sum_{i=1}^{p} \frac{d_i}{x_i^* - a} \right] \left[l_1 \sum_{i=1}^{p} d_i \log (y_i^* - a) - l_1 \sum_{i=1}^{p} d_i \log^2 (y_i^* - a) \right. \\ \left. + \left(\sum_{i=1}^{p} d_i \log (y_i^* - a) \right)^2 \right] - l_1^2 \sum_{i=1}^{p} \frac{d_i \log (y_i^* - a)}{y_i^* - a} = 0 \end{aligned} \tag{22}$$

6. SHORT-CUT PROCEDURES

In order to find a first approximation for a to be used in Eqn. (22), there is available a short-cut procedure which was first described by Gaddum [5]. Gaddum noted that if z_1, z_2, z_3 are three values from a log-normal distribution which are equidistant on the normal deviate scale then $z_2 - a$ is the geometric mean of $z_1 - a$ and $z_3 - a$. He has, therefore, suggested the following simple relationship for estimating a

$$a^* = \frac{z_1 z_3 - z_2^2}{z_1 + z_3 - 2z_2} \tag{23}$$

where a^* denotes the estimate of a.

The values of z_1, z_2, and z_3 may easily be obtained by plotting the cumulative distribution of the survival times, $\sum d_i/l_1$, on the log-normal graph paper discussed in the introduction. Then, taking z_2 to be the median of the sample distribution, the values on the time scale corresponding to any two points on the sample distribution which are a given distance, h, above and below the median on the normal deviate scale will be z_3 and z_1, respectively.

After a^* has been determined, c^*, the initial estimate of c to be used in Eqns. (18) and (19), may be found from the following relation:

$$c^* = \frac{\log (z_3 - a^*) - \log (z_1 - a^*)}{2h} \tag{24}$$

where $2h$ is the distance between the points corresponding to z_1 and z_3 on the normal deviate scale.

If desired, an initial estimate of b, b^*, may be easily obtained from the relation

$$b^* = z_2 - a^* \tag{25}$$

For known a, and for complete data the estimate c^* is most efficient when h is equal to 1.476, so that z_1 is equal to the 7th percentile, and z_3 is equal to the 93rd percentile of the sample distribution. The efficiency [3, pp. 77, 265] of this estimate is .65. The efficiency of the estimate b^* for known a is .637. The efficiency of the estimator a^* is not known.

7. RECAPITULATION

To recapitulate, the steps in the procedure discussed above for estimating the parameters a, b, and c, from the life table data are as follows:

1. Calculate the d_i using Eqns. (6), (7), and (8).
2. Plot $\sum d_i/l_1$ on log-normal graph paper.
3. Determine the median z_2, and, choosing h as close to 1.476 as feasible, the values z_1 and z_3.
4. Calculate a^* and c^* by Eqns. (23) and (24), respectively.
5. Determine t_2 and t_p by means of Eqns. (14) and (15), respectively, and use these values to calculate t_1^* and t_p^* from Eqns. (16) and (17), respectively.
6. Using the values obtained in Step 5, calculate $\log (y_1^* - a^*)$, $\log (y_p^* - a^*)$, y_1^* and y_p^* from Eqns. (18) and (19), substituting for a and c in these equations the values a^* and c^* obtained in Step 4.
7. For $i = 2, 3, \cdots, p-1$ calculate $\log (y_i^* - a)$ and y_i^* by means of Eqn. (13).
8. Solve Eqn. (22) for $\hat{a}$, using a^* and the values of y_i^* and $\log (y_i^* - a^*)$ obtained in Steps 6 and 7 as initial estimates. This is the most tedious step in the procedure, because in addition to the calculations prescribed by the equation, it will be necessary, for each successive estimate a_j^* of a, to make a new estimate c_j^* of c, and to calculate new y_i^*. Inverse interpolation can be used to determine $\hat{a}$ once a sufficiently small interval (a_r^*, a_s^*) has been found such that
$$\lambda(a_r^*) > 0 \quad \text{and} \quad \lambda(a_s^*) < 0.$$
9. The estimates $\log \hat{b}$ and $\hat{c}^2$ can be found from Eqns. (20) and (21) using the value of $\hat{a}$ calculated in Step 8 and the values of $\log (y_i^* - \hat{a})$ based on it.

8. GOODNESS-OF-FIT TESTS

The Chi-square test is commonly used to test for goodness-of-fit of an observed distribution function to a theoretical or calculated distribution function. The Chi-square test may also be applied on heuristic grounds to the present situation. A convenient expression, in terms of the life table data, is:

$$\chi_n^2 = \frac{N}{l_1} \sum_{i=1}^{p} \frac{(d_i - l_1 \cdot f_i)^2}{l_1 \cdot f_i} \tag{26}$$

where f_i is the expected relative frequency for the ith interval obtained by transforming the endpoints of the life table intervals into standard normal variates by means of Eqn. (10) and using a cumulative normal distribution table to obtain

$$f_i = \text{Prob}\ (t_i < t < t_{i+1}) \tag{27}$$

The number of degrees of freedom, n, is equal to $p-4$.

The value of N, what is commonly called the number of observations, is not uniquely determined in the present situation. Limits may be set on the value of X_n^2 by assigning an upper and lower bound to N, viz.

$$N' = L_1 \tag{28}$$

and

$$N'' = L_1 - \sum_{i=1}^{p} W_i, \tag{29}$$

respectively. The two values of X_n^2 thus obtained will yield lower and upper bounds on the value of P, the probability of obtaining a fit as bad or worse than that observed.

In the case of the study on chronic leukemia, it was decided to use the lower probability in deciding if a fit was too poor to be acceptable, while an intermediate value obtained by assigning to N the value

$$N''' = L_1 - \frac{1}{2}\sum_{i=1}^{p} W_i \tag{30}$$

was reported if the fit was clearly acceptable.

AN EXAMPLE

The following example was drawn from a recent investigation on the survival of patients with chronic leukemia [4]. The largest group, males with chronic lymphocytic leukemia, consisted of 234 individuals of whom 26 (11.1%) were lost to observation. The duration of survival of each patient was measured in months from the date of first diagnosis. The reasons for this choice have been adequately discussed elsewhere [8, 9]. The life table was calculated and the adjusted survival curve was plotted as shown in Table 1 and in Fig. 1.

Two estimates of "a" were made by the short-cut procedure. In the first, the twentieth, fiftieth, and eightieth percentiles were estimated from the graph. These yielded values of 2.2, 11.2, and 34.5 months, respectively, and the resulting value of a_1^* using Eqn. (23), was -3.5. In the second estimate, the fourteenth, fiftieth, and eighty-sixth percentiles were estimated from the graph. The corresponding values were 1.0, 11.2, and 41.8 months, respectively, giving a value of a_2^* of -4.1. This discrepancy is due to slight variations in the curvature of the observed distribution function. The values of b^* and c^* corresponding to the above estimates of a^* are shown in the accompanying table.

By means of Eqns. (14) and (15), and the values $l_1 = 1{,}000$, $d_1 = 142$, and

TABLE 1. COMPARISON OF SEVERAL METHODS OF FITTING A TRANSLATED LOG-NORMAL DISTRIBUTION TO THE OBSERVED SURVIVAL OF PATIENTS WITH CHRONIC LYMPHOCYTIC LEUKEMIA

Months after diagnosis	Number of deaths observed	Number of withdrawals	Expected number of deaths of an initial 1000 patients as calculated by				
			Life Table	Lea's Method	Short-cut Method		Modified maximum likelihood method
					20, 50, 80 percentiles	14, 50, 86 percentiles	
0	33	3	142	89	149	140	138
1	11	2	48	92	45	43	43
2	9	1	40	74	42	42	43
3	6	0	26	65	42	41	41
4	13	0	57	55	38	39	39
5	9	1	40	42	35	36	37
6	9	1	40	43	32	35	35
7	4	0	18	28	31	34	33
8	3	1	13	28	29	29	30
9	8	1	36	32	27	29	29
10	8	1	36	23	25	27	27
11	6	0	27	24	24	25	25
12– 15	12	0	55	53	62	66	67
15– 18	11	1	51	43	51	55	55
18– 21	4	0	19	31	42	46	45
21– 24	10	3	47	30	37	38	38
24– 27	10	0	48	21	30	32	33
27– 30	6	0	29	21	26	28	27
30– 33	4	1	20	16	23	23	23
33– 36	5	0	25	16	19	20	20
36– 42	8	4	42	23	33	33	33
42– 48	4	0	23	20	25	25	25
48– 54	4	0	22	14	19	19	19
54– 60	1	0	6 } 28	15 } 29	17 } 36	15 } 34	15 } 34
60– 72	5	4	32	18	23	23	23
72– 84	3	1	27	13	17	14	14
84– 96	1	0	10 } 37	11 } 24	12 } 29	11 } 25	11 } 25
96–120	0	0	0				
120–132	1	0	11 } 21	60	45	32	32
132+	0	1	10				
	208	26	1000	1000	1000	1000	1000
Estimates of parameters			a	0	−3.5	−4.1	−4.0
			b	8.3	14.7	15.3	15.2
			c	0.6799	0.4904	0.4418	0.4433
			d.f.	23	22	22	22
			$\chi^2_{min}(N=208)=$	47.53	19.33	16.99	16.68
			$P''=$	.002	.62	.76	.78
			$\chi^2_{max}(N=234)=$	53.47	21.74	19.12	18.77
			$P'=$	<.001	.48	.64	.66

$d_p = 10$, obtained from the life table, the values of t_2 and t_p were calculated. These were found to be -1.07 and 2.33, respectively. Solving Eqns. (16) and (17) gave values of t_1^* and t_p^* of -1.58 and 2.64, respectively.

Steps 6, 7, and 8 of the Recapitulation were then carried out using the values of t_1^*, t_p^*, a_1^*, and a_2^* calculated above. After just one iteration the interval (a_r^*, a_s^*) was sufficiently narrow to allow for inverse interpolation, and the value $\hat{a}$ equal to -4.0 was obtained. The estimates $\hat{b}$ and $\hat{c}$ were then calculated and are shown in the Table. The resulting distribution function is shown in Fig. 2.

For comparative purposes, the values obtained by Lea's method [7], where a is assumed *a priori* to be zero, are also shown. Goodness-of-fit tests were done for each set of parameters. The range on the value of χ^2 and P is shown for each test in the last four rows of Table 1. Since there were 26 intervals after pooling, the degrees of freedom using Lea's method are 23, while for the other methods they are 22.

It is seen that the best fit is obtained by the modified maximum likelihood method. It should be observed that the second set of estimates by the short-cut procedure yielded solutions that differed but slightly from the modified maximum likelihood solutions. This suggests that if the adjusted distribution function is sufficiently smooth and if the points z_1, z_2, and z_3 are properly chosen, then a great deal of computational labor may be avoided with only a slight loss in accuracy by using the short-cut procedure alone.

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ON THE CHOICE OF PLOTTING POSITIONS ON PROBABILITY PAPER

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This article is an attempt to throw more light on the problem of choosing plotting positions for the frequencies related to ordered sample values. The problem is particularly pertinent where special-scale graph paper is used such that a straight line indicates conformance of data to type of distribution assumed to apply. Such graph paper is often briefly referred to as "probability paper." The types of distributions here considered are the normal and the extreme-value distribution of Type I, with the accent on the extreme-value case.

Various methods of determining plotting positions have been put forward and there is confusion of thought in judging what plotting convention is optimum. An optimum criterion proposed by Chernoff and Lieberman has received wide attention. Blom in a recent book develops further criteria [2].

To clarify the problem the author has analyzed the purposes to be served in plotting data on probability paper. With a specific concept of the purpose to be served in mind, it becomes clearer to the investigator which plotting convention would be the best one to use.

1. GENERAL STATEMENT OF THE PROBLEM

LET $x_1, x_2, \cdots, x_n$ denote the sample values of a variate x which has location parameter μ and scale parameter β. Define the "reduced variable" y by

$$y = (x - \mu)/\beta. \tag{1}$$

The distribution functions of the reduced variable often do not involve unknown parameters. Two outstanding examples are the normal distribution

$$F(y) = (1/\sqrt{2\pi}) \int_{-\infty}^{y} e^{-t^2/2} dt \tag{2}$$

and the extreme-value distribution of Type I

$$F(y) = \exp\,[-e^{-y}]. \tag{3}$$

In the case of the normal distribution μ is the mean of x and β is its standard deviation, and in the case of the extreme-value distribution μ is the mode of x and β is $\sqrt{6}/\pi$ times its standard deviation.

Special coordinate plotting papers have been devised such that the x values are spaced arithmetically on the vertical scale and the y values are spaced similarly on the horizontal scale. There is further superimposed on the horizontal scale, a scale of cumulative frequency values obtained from relationships like (2) or (3) above, with values of $F(y)$ directly above or coincident with the corresponding y values (Chart 1).*

A property of such plotting paper is that if observed values of x conform to

* Admitted that the variable x in ordinary graphs is taken as the horizontal coordinate, most *probability paper* is designed to show the functional scale $F(y)$ measured horizontally. Thus the accompanying charts are so depicted.

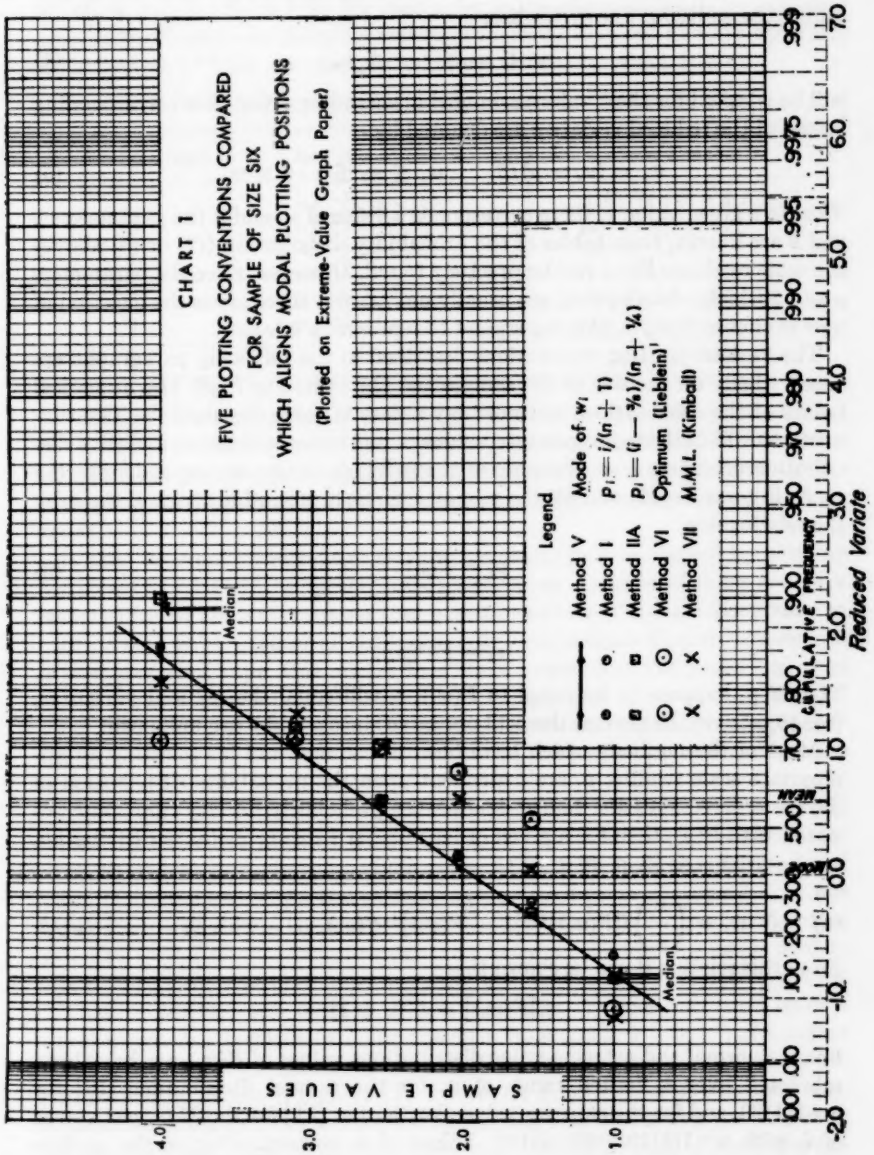
CHART 1
FIVE PLOTTING CONVENTIONS COMPARED
FOR SAMPLE OF SIZE SIX
WHICH ALIGNS MODAL PLOTTING POSITIONS
(Plotted on Extreme-Value Graph Paper)
Legend
Method V Mode of w_i
Method I $p_i = i/(n+1)$
Method IIA $p_i = (i - 3/8)/(n + 1/4)$
Method VI Optimum (Lieblein)'
Method VII M.M.L. (Kimball)
Median
Median
SAMPLE VALUES
4.0
3.0
2.0
1.0
MEAN
MODE
CUMULATIVE FREQUENCY
Reduced Variate
.001 .01 .100 .200 .300 .500 .700 .800 .900 .950 .980 .990 .995 .9975 .999
-2.0 -1.0 0 1.0 2.0 3.0 4.0 5.0 6.0 7.0

the assumed distribution function $F(y)$ the relation between x and $F(y)$ will be linear. The following discussion will be concerned with the x values arranged in order of magnitude. The symbolism

$$u_1 \leqq u_2 \leqq \cdots \leqq u_n \tag{4}$$

will be adopted for the x values arranged in ascending order. The corresponding values of the reduced variate y are denoted by

$$w_i = (u_i - \mu)/\beta, \qquad w_1 \leqq w_2 \leqq \cdots \leqq w_n. \tag{5}$$

Thus to a given value u_i there corresponds a value of w_i and if the parameters μ and β are known, from tables of the probability distribution ((2) or (3) above) the scale readings $F(w_i)$ can be precisely found. However, since the parameters μ and β which relate w_i to u_i are usually not known, there arises the question of how to choose $F(w_i)$ in plotting a point to represent a u_i value.

The various plotting conventions described in the following paragraphs are based wholly on the size of the sample and the nature of $F(y)$. The numerical values of the observations will not play a part in these descriptions. The conventions will indicate appropriate values p_i on the scale of the cumulative distribution or values v_i on the scale of the reduced variate corresponding to the p_i. As in formulas (2) and (3) the p_i may then be expressed in terms of the v_i by the relationship

$$p_i = F(v_i) \tag{6}$$

or conversely

$$v_i = F^{-1}(p_i).$$

Thus if p_i happens to be equal to $F(w_i)$, v_i will equal the reduced variate w_i (see (5) above). In general this will not be so since μ and β are unknown.

Most of the methods which have been in common use for selecting an appropriate scale reading p_i are dependent upon the probability distribution of the ith ordered statistic. If $f(y)$ denotes the probability density function associated with $F(y)$, then for a given order number i this probability distribution can be written as [2, p. 40 or 6].

$$pdf \quad \text{of} \quad w_i = \binom{n}{i} i[F(w_i)]^{i-1}[1 - F(w_i)]^{n-i} f(w_i), \qquad \text{with } w_i = (u_i - \mu)/\beta.$$

From this relation it can be shown that

$$E[F(w_i)] = i/(n + 1).$$

For the normal and extreme-value distributions values of $E(w_i) = \omega_i$ have been tabulated over a limited range of n; for the normal distribution [13] for $n = 1(1)20$, and for the extreme-value distribution [11] for the 25 largest values of i, with $n = 1(1)10(5)60(10)100$. Values of p_i corresponding to the median and mode of w_i have been recommended by some [1, 5, 6]. Median values have been tabulated for $n = 1(1)10$ [15]. Modal values of w_i have not been tabulated. One early method [7] recommended is to take $p_i = (i - \frac{1}{2})/n$. This amounts to choosing the middle value of the interval from $(i-1)/n$ to i/n. Thus at the

extremes $p_1 = \frac{1}{2}/n$ and $p_n = (n-\frac{1}{2})/n$ designate the scale readings to be associated with u_1 and u_n. Recently Blom [2] has recommended for the normal distribution, $p_i = (i-\frac{3}{8})/(n+\frac{1}{4})$. We shall refer to the above methods for selecting p_i as "direct" methods since the rationale involved is based directly upon the order number i.* They are summarized in the tabulation which follows.

DIRECT METHODS OF SELECTION OF SCALE READINGS p_i

Method Number	Reduced Variate v_i	Scale Reading p_i
I	$v_i = F^{-1}[i/(n+1)]$	$p_i = i/(n+1) = E[F(w_i)]$
IIA	$v_i = F^{-1}[(i-\frac{3}{8})/(n+\frac{1}{4})]$	$p_i = (i-\frac{3}{8})/(n+\frac{1}{4})$
II	$v_i = F^{-1}[(i-\frac{1}{2})/n]$	$p_i = (i-\frac{1}{2})/n$ (7)
III	$v_i = E(w_i) = \omega_i$	$p_i = F(\omega_i)$
IV	$v_i =$ median of $w_i = \zeta_i$	$p_i = F(\zeta_i)$
V	$v_i =$ mode of $w_i = \epsilon_i$	$p_i = F(\epsilon_i)$

2. THE PURPOSE SERVED IN USING PROBABILITY PAPER

Before proceeding further it is to be noted that the simplification afforded by the use of probability-scale graph paper is a *visual* simplification. The probability paper transforms a curvilinear distribution into a straight line. If the approach is to be purely analytical there is no point in using the special scale paper.

It then becomes important to have in mind the purpose served by plotting the observed data on the special scale graph paper. In general there are three rather different kinds of purposes that might be served. These are:

(1) A test as to whether or not the sample data indicate that the universe is of the prescribed type. One argues that the universe is of the prescribed type only if the plotted points tend to lie along a straight line.

(2) The graphical method may be used as a short-cut in estimating the standard deviation of the distribution, which in turn is directly determined from the *slope* of the fitted straight line.

(3) Graphical extrapolation at one of the extremes. This is the purpose most commonly served in plotting data from an extreme-value universe. Data are often plotted on extreme-value graph paper when it is known that on the lower range of maximum values the extreme-value distribution of Type I does not apply [9, p. 767]. If in the upper range there is likely to be good conformity with the Type I distribution a straight line fitted on the upper range is used as a basis for extrapolation of large extremes beyond the range of plotted points.

These objectives can overlap. When accenting (2) one may well have a weather eye on Objective (1). For example, in examining a batch of samples taken from different populations one may have inferred that the universes are normal and so accent Objective (2). However, some errant populations may deviate considerably from the normal and so one may also give some weight to the graphical test of normality. Similarly in accenting Objective (3), Objective (2) is important. Furthermore, as noted above, there are situations where the

* In Method IIA recommended by Blom for the normal distribution a criterion not directly related to the order number is included [2, p. 144 (c)]. Since the convention closely follows Method II we have included it under the category of "direct" methods.

data over the lower range are known not to follow the prescribed distribution; in which case data on the upper range are given greater weight in fitting a straight line. Thus Objective (1) is involved in indicating what part of the range conforms to the prescribed universe.

3. CRITERION SUGGESTED BY CHERNOFF AND LIEBERMAN

Chernoff and Lieberman [3, 4] have suggested a criterion for selecting in the normal case an "optimum" method of plotting.[1] In the earlier paper the authors state:

> "The purpose of this paper is to indicate, with a special example, that any graphical technique should depend to a large extent on the purpose for which the graph is drawn. In particular, it presents tables covering sample sizes up to 10, for selecting the (scale readings p_i) on normal probability paper so as to obtain 'optimum' graphical estimates of the mean and the standard deviation σ of a normal population. The somewhat more complicated problem of selecting the (scale readings p_i) to obtain an 'optimum' test for normality is not discussed."[2]

Clearly *CL* do not pretend to offer a criterion for Objective (1), and further warn that the purpose served by the graph should be a factor in making a selection of methods.

In order to translate the process of visually fitting a straight line to a set of plotted points *into an analytical process* *CL* propose the assumption (hereinafter referred to as Assumption *A*): *Assumption A* "Let us assume that the visually fitted line is a very good approximation to the line that would be obtained by minimizing the sum of squares of the deviations (in the x direction) from the line."

This assumption quoted from the 1954 paper is also the basis of the analysis further pursued in the 1956 paper. The analytical process of fitting a line to a prescribed set of plotting positions which minimizes the sum of squares of the deviations in the x direction yields sampling variances of the resulting analytical estimates of the parameters μ and β. Thus *CL* argue that the optimal character of a set of plotting positions might be judged in terms of the resulting magnitudes of the sampling variances of the parameters. This is taken as their criterion for judging the optimal character of a plotting method.

Since the parameter β, determined from the slope of the line, is the more difficult one to estimate, the accent is placed on the variance of this estimate. Thus *CL* in the 1954 paper determine values of p_i for sample sizes 2 to 10 which minimize the variance of an unbiased estimate of $\beta = \sigma$. Also it is shown that a biased estimate of β with minimum mean square deviation can be found, and the plotting positions for this biased estimate of β are given.

In the 1956 paper *CL* also prove that for the normal distribution selection of p_i based on minimizing the variance of β also furnishes the optimum plotting positions for purposes of extrapolation [4, Theorems 1 and 2, and Example 2].

The validity of the *CL* criterion and of the corollary just stated depends upon the validity of Assumption *A*.

The *CL* criterion has received considerable attention because it provides analytical tests for judging the optimal character of a plotting method.

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[1] Future reference to these writers will be by the letters *CL*.

[2] Parenthesis supplied by present writer.

Blom in his 1958 book [2, p. 144, conditions (a) to (c)] includes the *CL* criterion, but also adds another condition; namely, that

Condition B "The points $(u_i p_i)$ lie on the average on a line which deviates only very little from a straight line." Analytically, Blom points out, this is to say that

$$E(u_i) = \text{approx. } \mu + \beta F^{-1}(p_i) = \mu + \beta v_i. \tag{8}$$

Note that this relation becomes *an equality* under Method III since in this case v_i is taken as $E(w_i) = \omega_i$.

We shall show that Assumption *A* upon which the *CL* criterion is based is much more plausible if the distribution of the universe is symmetrical than if otherwise. Since the extreme-value distribution is finding a wide application in many scientific fields, we shall examine the application of the *CL* criterion to this type of distribution both from the standpoint of the numerical results which flow from accepting the *CL* assumption, and from the point of view of their relation to the graphical purpose which is to be served.

4. EXTENSION OF THE *CL* ANALYSIS TO THE EXTREME-VALUE DISTRIBUTION

As noted under (3) above, for the extreme-value distribution μ is the mode of the parent distribution. Since the standard deviation of the reduced variate y is known to be $\pi/\sqrt{6}$ it follows that the scale parameter β is related to the standard deviation of x by

$$\beta = \sigma_x \sqrt{6}/\pi.$$

Thus the role of β in determining the best fitting line to a set of plotted points (v_i, u_i) on extreme-value paper is the same as that of σ in the problem treated by *CL* in the 1954 article. Applying least squares' theory (minimizing sum of squares of deviations in the x direction) the equation of the fitted straight line $u = \hat{\mu} + \hat{\beta} v$ satisfies the normal equations

$$\hat{\beta}\bar{v} + \hat{\mu} = \bar{u}$$

$$\hat{\beta} = \Sigma c_i u_i, \qquad c_i = \frac{v_i - \bar{v}}{\Sigma (v_i - \bar{v})^2}, \qquad \Sigma c_i = 0. \tag{9}$$

We have here an estimate of β based on a linear function of the observed values u_i. Optimum values of the coefficients c_i have been determined by Lieblein to give an unbiased estimate of β with minimum variance for samples up to size six [12]. Another choice of these coefficients yielding an unbiased estimate of β for samples of sizes extending to $n = 112$, based on a modification of the maximum likelihood equations has been suggested by the author in a previous paper [9].

It will be of interest to translate these methods by means of the equations (9) into conventions for choosing sets of plotting points (v_i, u_i) and note the results on extreme-value plotting paper. This we shall do for a sample of size six since this is the largest sample for which the optimum coefficients have been determined. The results will be compared with the direct conventions for choice of plotting positions v_i indicated in (7) above.

Before discussing the determination of v_i from c_i, we present equations for evaluating the bias in the resulting estimates of μ and β based on (9). Keeping in mind the fact that v_i, $\bar{v}$ and c_i are predetermined by the particular plotting convention used, we have

$$E(\hat{\mu}) = E(\bar{u}) - \bar{v}E(\hat{\beta})$$
$$E(\hat{\beta}) = \Sigma c_i E(u_i).$$

With w_i denoting the true value of the reduced variate associated with u_i

$$u_i = \mu + \beta w_i$$
$$\bar{u} = \mu + \beta\bar{w}$$

and

$$E(\bar{u}) = \mu + \beta E(\bar{w}) = \mu + \beta E(y) = \mu + \beta\gamma,$$

where

$$\gamma = \text{Euler's constant} = 0.577216 \cdots.$$

For a specific order index $i = m$, holding m constant,

$$E(u_m) = \mu + \beta E(w_m) = \mu + \beta\omega_m.$$

Substituting in the first two equations, the biases in μ and β are given by

$$E(\hat{\mu}) - \mu = \beta[\gamma - (E(\hat{\beta})/\beta)\bar{v}]$$
$$E(\hat{\beta}) - \beta = \beta[\Sigma c_i\omega_i - 1]. \tag{10}$$

These biases can be numerically computed for a limited range of n, except for the factor β since γ is known and ω_i has been tabulated [11].

The determination of the plotting positions v_i from c_i, which are consistent with (9) now proceeds as follows. From the second of the equations (9) one finds that

$$\Sigma(v_i - \bar{v})^2 = 1/\Sigma c_i^2.$$

Hence $(v_i - \bar{v})$ is determined by the second equation. Choice of $\bar{v}$ is *independent* of such determination. However, the *bias* of μ as indicated by (10) depends upon the choice of $\bar{v}$. Each of the two sets of c_i mentioned above give *unbiased* estimates of β. Hence to give an unbiased estimate of μ we take $\bar{v} = \gamma$. With $\bar{v}$ thus chosen, the plotting positions v_i are completely determined from the second equation of (9) to give unbiased estimates of β and μ.

Conversely with values of v_i taken from the *direct* conventions indicated by (7) values of c_i are determined from (9) with $\bar{v}$ determined as the mean value of the v_i selected. The resulting biases in $\hat{\mu}$ and $\hat{\beta}$ are then found from (10).

In order to compute the variance of an estimate of β note that since $\Sigma c_i = 0$,

$$\hat{\beta} = \Sigma c_i(\beta w_i + \mu) = \beta\Sigma c_i w_i \tag{11}$$

where w_i denotes the true value of the reduced variate associated with u_i. Since Lieblein has determined the variance-covariance matrix of w_i for a sam-

ple of size six [12, Table B-1] the sampling variances of the various estimates of β by the linear relation (9) can be computed.

In the tables included in Schedule A are shown the pertinent data associated

SCHEDULE A. DATA FOR APPRAISAL OF OPTIMUM CHARACTER OF SEVEN PLOTTING CONVENTIONS BASED ON CRITERION PROPOSED BY CHERNOFF AND LIEBERMAN APPLIED TO EXTREME-VALUE SCALE PLOTTING PAPER FOR SAMPLE OF SIZE SIX

(Column headings indicate basis of plotting convention)

Order No. i	I $p_i = i/(n+1)$	IIA $p_i = \frac{i-\frac{1}{2}}{n+\frac{1}{2}}$	II $p_i = (i-\frac{1}{2})/n$	III $v_i = \omega_i$	IV $v_i = \zeta_i$	V $v_i = e_i$	VI Optimum (Lieblein)	VII *M.M.L.* (Kimball)
TABLE 1. PLOTTING VALUES OF VARIABLE v_i								
1	− .6656	− .8340	− .9102	− .7773	− .7955	− .8284	−1.0793	−1.1620
2	− .2254	− .2979	− .3266	− .2545	− .2850	− .3440	0.4474	0.0373
3	.1658	.1421	.1330	.1884	.1460	.0651	0.8412	0.6374
4	.5804	.6075	.6180	.6627	.6030	.4902	1.0342	1.0368
5	1.0893	1.2003	1.2459	1.2750	1.1805	1.0059	1.1164	1.3366
6	1.8703	2.2504	2.4417	2.3690	2.1583	1.7919	1.1033	1.5772
Mean of v_i	.4691	.5114	.5336	.5772	.5012	.3635	.5772	.5772
TABLE 2. PLOTTING VALUES OF SCALE READINGS p_i ON EXTREME-VALUE PAPER								
1	.1429	.1000	.0833	.1135	.1091	.1013	.0527	.0409
2	.2857	.2600	.2500	.2753	.2644	.2440	.5277	.3816
3	.4286	.4200	.4167	.4368	.4214	.3918	.6497	.5894
4	.5714	.5800	.5833	.5972	.5786	.5420	.7008	.7015
5	.7143	.7400	.7500	.7562	.7356	.6937	.7208	.7689
6	.8571	.9000	.9167	.9107	.8909	.8465	.7176	.8134
TABLE 3. COEFFICIENTS e_i RELEVANT TO LEAST SQUARES' FIT								
1	− .2688	− .2202	− .2022	− .2122	− .2298	− .2661	− .4593	− .3383
2	− .1645	− .1325	− .1205	− .1303	− .1393	− .1579	− .0360	− .1050
3	− .0718	− .0604	− .0561	− .0609	− .0629	− .0666	.0732	.0117
4	.0264	.0157	.0118	.0134	.0180	.0283	.1267	.0894
5	.1469	.1128	.0998	.1093	.1204	.1434	.1495	.1477
6	.3319	.2846	.2672	.2807	.2936	.3189	.1458	.1944
TABLE 4. BIAS AND VARIANCE OF THE ESTIMATES WHICH RESULT FROM LEAST SQUARES' FIT[a]								
Bias of $\hat{\mu}$	.0010	.0546	.0728	0	.0433	.1441	0	0
Bias of $\hat{\beta}$	.2284	.0220	− .0547	0	.0632	.1916	0	0
Variance of $\hat{\beta}$	.2489	.1754	.1513	.1686	.1888	.2325	.1320	.1384
(Bias)^2	.0522	.0004	.0030	0	.0040	.0367	0	0
Mean Squ. Dev.	.3011	.1758	.1543	.1686	.1928	.2692	.1320	.1384
TABLE 5. VARIANCE OF ESTIMATES CORRECTED FOR BIAS[a]								
Cor. Factor	.8141	.9785	1.0579	—	.9406	.8392	—	—
Squ. of Cor. Factor	.6628	.9574	1.1192	—	.8846	.7043	—	—
Var. of Cor. Est.	.1650	.1679	.1693	—	.1671	.1637	—	—

[a] Biases of $\hat{\mu}$ and $\hat{\beta}$ are to be multiplied by factor β, not shown (see equation (10)). Variance and mean square deviation of $\hat{\beta}$ are to be multiplied by factor β^2, not shown (see equation (11)).

with eight plotting conventions. As indicated previously the order number i proceeds from the smallest x_i to the largest. The headings tell the basis of each plotting convention and are self-explanatory except that *M.M.L.* stands for the "modified maximum likelihood" method of determining the coefficients in Table 3. As mentioned in the footnote the figures for biases and variances involve the factors β and β^2, respectively. This follows from (10) and (11).

Where the bias of $\hat{\beta}$ is not removed by a correction factor the mean square deviation from the true value is shown along with the variance (Table 4). In Table 5 are shown the variances of estimates corrected for bias. This table is important if the graphical method were to be supplemented by introducing a correction for bias. Note that the correction factor is obtained by dividing β by $E(\hat{\beta})$ and that this ratio is *independent* of the value of β, and can be computed for any value of n for which ω_i has been tabulated [11, 13]. Furthermore it brings out the fact that the "direct" methods when corrected for bias, all have about the same variance. The same phenomenon applies in the normal case (Table 10 of Schedule B). Furthermore a positive bias produces a large mean square deviation, while a negative bias produces a small one (cf. Methods I and II). This brings up the question as to whether or not the "bias" should be equalized before measuring the optimal character of a plotting convention by its mean square deviation.

Note. As a digression from the problem posed in this paper one may note in connection with Methods VI and VII—the optimum and *M.M.L.* choice of c_i—that the variance of the estimate of β by Method VII is only *slightly* greater than that of Method VI. This may perhaps indicate that the same is true for larger samples. This is of particular interest in connection with the problem of obtaining the best estimate of β based on a linear function of the observed sample values u_i because the *M.M.L.* method involves coefficients which are easily computable for sizes of samples up to $n=112$, while the optimum choice of the c_i is only available for sample sizes up to $n=6$ [9].

For purposes of comparison we present in Schedule B the data associated with the *CL* criterion as it is applied to a *normal* universe, for the same sized sample. Method VI refers to the linear unbiased estimate of minimum variance referred to as Method 2 in Table II of the *CL* 1954 paper, and Method VII refers to the linear *biased* estimate of minimum mean square deviation, referred to as Method 4 in that table. The biases of β have been computed directly from the formula (10) above. It will be noted that because of the symmetry of the normal distribution there is *no bias* in the estimates of μ, and it is necessary to show only plotting positions on one side of the mean. The others are obtained from considerations of symmetry. In Tables 6 and 8 this means a reversal of sign, and in Table 7 the complimentary figure is required.

As in Schedule A it is of interest to note from Table 10 that when estimates of β are corrected for bias there is little to choose between the direct methods, judging by the magnitudes of the variances of the estimates; and in the normal case the indirect methods also have very nearly the same variance.

5. DISCUSSION OF AN EXAMPLE

We now consider an example of what happens when the plotting positions p_i indicated by Table 2 of Schedule A are used for plotting data on extreme-

SCHEDULE B. DATA ASSOCIATED WITH *CL* CRITERION APPLIED TO NORMAL PROBABILITY PAPER FOR SAMPLES OF SIZE SIX

(Column headings indicate basis of plotting convention)

Order No. i	I $p_i = i/(n+1)$	IIA $p_i = \frac{i-\frac{1}{2}}{n+\frac{1}{2}}$	II $p_i = (i-\frac{1}{2})/n$	III $v_i = \omega_i$	IV $v_i = \xi_i$	V $v_i = \epsilon_i$	VI Unbiased Linear (*CL*)	VII Biased Linear (*CL*)
TABLE 6. PLOTTING VALUES OF VARIABLE v_i								
4	.1800	.2019	.2104	.2016	.1983	.1919	.1773	.1960
5	.5660	.6433	.6745	.6418	.6297	.6059	.5684	.6285
6	1.0676	1.2816	1.3830	1.2672	1.2313	1.1601	1.3025	1.4402
TABLE 7. PLOTTING VALUES OF SCALE READINGS p_i ON NORMAL PROBABILITY PAPER								
4	.5714	.5800	.5833	.5799	.5786	.5761	.5704	.5777
5	.7143	.7400	.7500	.7395	.7356	.7277	.7151	.7352
6	.8571	.9000	.9167	.8975	.8909	.8770	.9036	.9251
TABLE 8. COEFFICIENTS c_i RELEVANT TO LEAST SQUARES' FIT								
4	.0603	.0481	.0436	.0490	.0508	.0548	.0432	.0391
5	.1896	.1534	.1398	.1559	.1613	.1731	.1386	.1253
6	.3577	.3056	.2867	.3078	.3154	.3315	.3175	.2872
TABLE 9. BIAS AND VARIANCE OF THE ESTIMATES OF σ WHICH RESULT FROM LEAST SQUARES FIT[a]								
Bias of $\hat{\sigma}$	.1741	− .0093	− .0763	0	.0269	.0845	0	− .0956
Variance	.14599	.10379	.09024	.1058	.1116	.1245	.10571	.08646
(Bias)^2	.03031	.00009	.00582	0	.0007	.0071	0	.00914
Mean Squ. Dev.[b]	.17630	.10388	.09605	.1058	.1124	.1316	.10571	.09560
TABLE 10. VARIANCE OF ESTIMATES CORRECTED FOR BIAS[a]								
Cor. Factor	.8517	1.0094	1.0826	—	.9738	.9221	—	1.1057
Squ. of Factor	.7254	1.0188	1.1721	—	.9484	.8504	—	1.2226
Var. of Cor. Estimate	.1059	.1058	.1058	—	.1059	.1058	—	.1057

[a] Bias of $\hat{\sigma}$ is to be multiplied by factor σ, not shown (see equation (10)). Variance and mean square deviation of $\hat{\sigma}$ are to be multiplied by factor σ^2, not shown (see equation (11)).

[b] Values of mean square deviation for Methods I, II, VI and VII have been taken from Table II of *CL* [3]. The values for Methods IIA, III, IV and V were computed by the author.

value scale plotting paper. For this purpose we use the sample most likely to be drawn; namely, the set $(\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_6)$ where $\tilde{u}_i = \mu + \beta\epsilon_i$.* The set of values $u_1 = 1.000$, $u_2 = 1.555$, $u_3 = 2.023$, $u_4 = 2.510$, $u_5 = 3.100$, $u_6 = 4.000$ with $\mu = 1.9484$ and $\beta = 1.1449$ satisfies this condition.

On Chart 1 we show the positions of four plotting conventions other than those of the modal w_i (which lie along a straight line). The plotted points marked by the small circles represent positions determined by Method I and those marked by squares represent Method IIA. It will be noted that the me-

* Some will argue that several randomly chosen samples would be preferable. The writer has set up such samples. It would be still better to obtain 100 such samples and summarise. Since the samples are from a known type of universe such a summary can best be made by using *known* properties of the universe which spawns them. This we have done by choosing the modal sample for examination. We hasten to point out that to be consistent, whatever *conclusions* are drawn must also be obtained from the known properties of the *parent* distribution rather than from the particular sample exhibited.

dian positions at the two extremes lie between the points plotted by Methods I and IIA. However, the positions of Method VI, determined indirectly from the optimum linear estimate of β (marked by large circles) and Method VII, determined from the modified maximum likelihood estimate of β (marked by X's), show a trend of a decidedly different character. The characteristic of this trend is that at both extremes the optimum indirect conventions show plotting positions at values of p *less* than the values associated with the central tendency (mode, mean of p_i, median, mean of w_i, etc; see Tables 1 and 2 for conventions not shown on the chart), and at the center of the set of values of the sample, the values of p_i are greater than values represented by the central tendency. Hence it may be inferred that *in general* for the extreme-value case samples plotted by Methods VI and VII will be represented by plotting positions which tend to lie along the above type of curve, rather than along a straight line.

Note that the curvilinear tendency just observed results from a systematic violation of Condition B introduced by Blom (see (8) above). From Table 1 of Schedule A we find that the v_i of Methods VI and VII are considerably less than the v_i of Method III at each extreme, and in the middle portion are greater than the v_i of Method III. Indeed, in the case of Method VI it is impossible to find a sample such that points plotted by this convention would lie along a straight line, since by this convention p_n is less than p_{n-1}.

Some of the reasons why the indirect conventions produce the distortion noted are quite apparent: (1) the over-all minimizing condition of least squares gives no recognition to the individual *signs* of the expected deviations other than that collectively the signs should be such as to produce an unbiased slope and mean (or modal) value; (2) since the variances of u_i are progressively larger as one proceeds towards the largest extreme, the numerical values of the coefficients c_i which produce minimum variance (cf. equation (11)) tend to be smaller at the upper extreme and larger at the lower extreme, than in the case of the direct conventions. This in turn steepens the trace of plotted points at the upper extreme and flattens it at the lower extreme.

On the other hand with a normal universe and points plotted on normal probability-scale plotting paper, the results are quite different. This is illustrated on Chart 2. Here again we have chosen a modal sample, with $\bar{u}_i = \mu + \beta\epsilon_i$. The set of values $u_1 = 1.000$, $u_2 = 1.717$, $u_3 = 2.252$, $u_4 = 2.748$, $u_5 = 3.283$, $u_6 = 4.000$ with $\beta = \sigma = 1.2930$ and $\mu = 2.500$ satisfies this condition. The corresponding readings on the probability scale are shown in Table 7 of Schedule B, under Method V. The small circles on the chart represent plotting positions selected by Method I and the crosses show plotting positions selected by Method II. The large circles indicate "optimum, unbiased" plotting positions recommended by *CL* in the 1954 paper,—Method VI of Schedule B. In order to more clearly indicate the central tendency at the extremes, we show the plotting positions of the median $p_i = F(\zeta_i)$ by arrows at $i = 1$ and $i = 6$,—Method IV of Schedule B.

It is notable that *in the normal case* the indirect method produces a trace of plotting positions quite in harmony with plotting positions following the central tendency. Of course, for large samples the spread at the extremes will be greater.

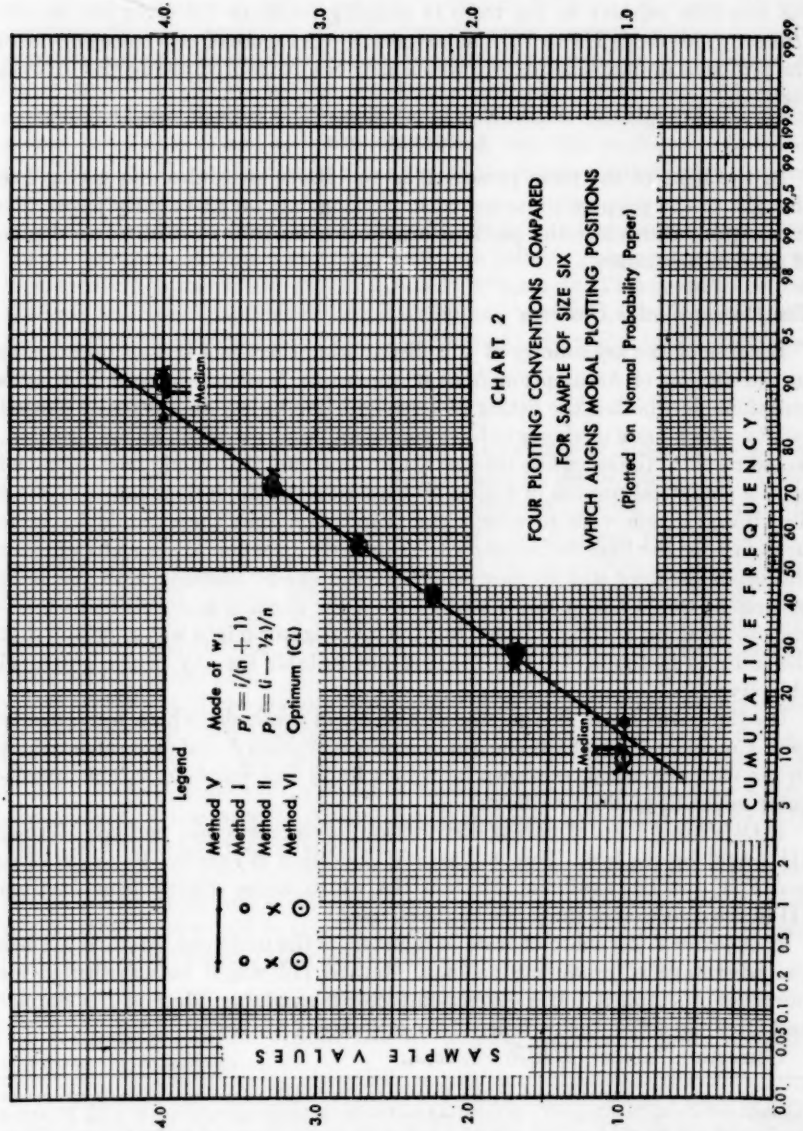
CHART 2
FOUR PLOTTING CONVENTIONS COMPARED
FOR SAMPLE OF SIZE SIX
WHICH ALIGNS MODAL PLOTTING POSITIONS
(Plotted on Normal Probability Paper)
Legend
Method V Mode of w_i
Method I $P_i = i/(n+1)$
Method II $P_i = (i - \frac{1}{3})/n$
Method VI Optimum (CL)
Median
Median
SAMPLE VALUES
CUMULATIVE FREQUENCY
4.0
3.0
2.0
1.0
0.01 0.05 0.1 0.2 0.5 1 2 5 10 20 30 40 50 60 70 80 90 95 98 99 99.5 99.8 99.9 99.99

But for any size sample, because of the symmetry of the normal universe, we shall not have the bending of the trace of plotting positions of Method VI in *one* direction relative to the trace of plotting positions following the central tendency (Methods III to V). The reversal in curvature of this trace alleviates the principal undesirable feature which may result from applying the indirect methods.

6. CONCLUSIONS

In the light of the facts presented in the above schedules and giving due thought to the purpose to be served in plotting data on probability paper, the investigator can select the plotting convention he will use with some degree of rational assurance.

The Extreme-Value Universe

Because of the asymmetry of this distribution there is a serious question as to the *validity* of Assumption A as it applies to Methods VI and VII. It is questionable whether the fitting of a straight line by eye to a trace of plotted points, which has a tendency to have considerable curvature all in one direction, would result in the same line (or nearly the same line) that would be determined by the analytical process of fitting by least squares (deviations taken in the x direction). If one were to accept the proposition that there would be some tendency for the lines to be nearly the same, the question arises as to whether the probable error due to uncertainty of the eye-fit agreeing with the least squares fit would not be much larger than if the plotting positions had a tendency to lie along a straight line. Thus the investigator might well conclude that the indirect methods, VI and VII, are not suitable for any of the graphical objectives.

On the other hand, in the case of the direct methods (which more nearly meet Condition B) perhaps one can accept Assumption A and give some weight to the variance measures shown in Tables 4 and 5 of Schedule A. The investigator might be guided as follows:

If *Objective 1* (test of parent distribution) is the main one, perhaps Method III would be the best. This will satisfy Condition B exactly. If values of ω_i and $F(\omega_i)$ are not available, Method IIA which varies slightly from Method III in the direction of the mode might be used.[1]

If *Objective 2* (estimate of scale parameter) is the main one, Method III has the advantage of producing no bias. Method IIA might be substituted for purposes of simplicity. If a correction factor is available, Method I is simpler and after correction the variance of the estimate of β is low.

Correction Factor $= \beta / E(\hat{\beta}) = 1/\Sigma c_i \omega_i$.

[1] Using some of the arguments used by Blom for proposing Method IIA as most suitable for the normal case, one might arrive at the formula $p_i = (i - \frac{1}{2})/(n + \frac{1}{2})$ as a good method in the extreme-value case [2, p. 74]. The author experimented with this and found that although it reduced the negative bias of $\hat{\beta}$ of Method II from $-.0547$ to $-.0266$, with *MSD* of .1630, that when corrected for bias the variance came to .1713. Method IIA incurs a slight positive bias of $\hat{\beta}$ of .0220, but variance of corrected estimate is at .1679. Furthermore the plotting positions are slightly to the right of ω_i, while in Method II*A* they are to the left of ω_i. Thus for IIA they are on the same side as the modal positions ϵ_i and hence on the side where probability density is greater. Accordingly the author has preferred to introduce Method IIA as the compromise between Methods I and II in the extreme-value case.

If *Objective 3* (extrapolation to the right) is the main one, there will be room for considerable exercise of judgment depending upon the weight given to the extremes u_{n-1} and u_n in fitting the line. Although Method III, under the presumption that Assumption A is valid, gives no bias some consideration might be given to the fact that the distribution of u_n is highly skewed. It might be well to supplement Method III (or the simpler Method IIA) by plotting the points determined from Method I. These latter are slightly to the right of the modal points (see Table 1) and so when taken along with the positions determined from Methods III or IIA furnish some idea of the range of uncertainty in extrapolating from the graph.[2]

In other words, in dealing with Objective 3, although measures found from the *CL* analysis, as shown in Schedule A, are helpful, judgment will have to be exercised that cannot be guided precisely by formula. This is apt to be particularly true for the extreme-value distribution, because at the lower extreme the sampled universe in many cases is known not to be precisely that described by formula (3).

The Normal Universe

In this case we do not seriously question the validity of Assumption A. Our greatest concern is with the matter of the "bias" in the estimate of $\beta = \sigma$. We are inclined to be suspicious of basing a criterion for optimal choice upon the mean square deviation, if a significant bias is present. Blom has introduced a formula, Method IIA, which pretty well meets such objection, and at the same time carries a low *MSD* of the estimate of β. It also is believed to satisfy Condition B for large values of n as well as for the values up to $n = 15$ which have been tested empirically [2, p. 71, Table 2]. To meet either Objectives 1 or 2 we recommend Method IIA.

For *Objective 3*, if one accepts Assumption *A* as entirely valid, we note that *CL* have shown in their 1956 paper [4] that the optimum criterion of minimum sampling variance (as computed in Table 9 above) or mean square deviation, if there is a bias, points to optimum extrapolation as well, in the same sense.* Accordingly, if one accepts Assumption A one can agree that Method IIA would be satisfactory for extrapolation.

If, however, the graphical fitting of the line is presumed to be more influenced by the plotting positions of the extremes (say at the right, with extrapolation to the right) than would be true under Assumption A, then again there is need for caution in relying upon any single formula.

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I am indebted to Dr. Joseph Lev for helpful suggestions toward greater clarity in the presentation.

[2] An interpolation chart which shows the position of the interval of maximum probability density for probability level .50 applicable to u_n for various values of n will prove helpful to the investigator faced with graphical extrapolation where considerable weight is given to the plot of the largest sample value. The author has set up such a chart both for the normal and the extreme-value universe which may be made available through publication at some time in the future.

* The author has proved, in an unpublished communication to Chernoff, 9-7-57, that the condition which leads to this corollary is not satisfied in the case of the extreme-value distribution.

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BIBLIOGRAPHY ON SEQUENTIAL ANALYSIS*

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This article contains 374 references dealing with the subject of sequential analysis. These references are listed alphabetically by author and also broken down into fourteen subject categories.

The concepts of sequential analysis go back many years but, in the modern sense of Wald, it is a relatively new subject barely having reached its fifteenth birthday. Nevertheless, a considerable amount of material has been written on various aspects of the subject. A great many of these articles have appeared in journals not published in this country and it would seem that a bibliography on this subject would be quite useful to those now working in the field. The format of this bibliography will be quite similar to that compiled by I. R. Savage (*Journal of the American Statistical Association*, Vol. 48, 1953, pp. 844–906) in the field of non-parametric statistics and will include, in so far as possible, all such articles which have appeared through 1959.

As in many fields of statistics, there is the problem of definition. Theoretically any procedure wherein the decision to gather more information depends on the information already gathered can be considered a sequential procedure. This bibliography, however, will be restricted primarily, though not exclusively, to the Wald concept of sequential analysis and in so doing will exclude such special cases as double sampling, multiple sampling and response surface techniques except for some of the basic pioneering works on these subjects. These are, admittedly, borderline cases. There are so many articles on these subjects, particularly in various industrial journals, that it would be impossible to cover them all; however, the basic references are given as well as any which are more directly tied into standard sequential techniques. A true sequential procedure can be considered as both a decision procedure and a stochastic process and as a result, there is considerable overlap in the content of the literature in these fields. Inasmuch as possible, this bibliography is restricted to those articles dealing with decision functions and stochastic processes which make direct reference to sequential analysis as such.

As in the case of Savage's article, entries in this bibliography are arranged alphabetically by author (first author only in the case of joint authorship) and chronologically within authors. The letters after each entry refer to the following list of categories:

A—General and expository articles.
B—Theory of sequential analysis.
C—Sequential procedures for the binomial distribution.
D—Sequential procedures for other discrete distributions.

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E—Sequential procedures for measures of central tendency—normal distribution.

F—Sequential procedures for measures of dispersion—normal distribution.

G—Sequential experimentation and sequential analysis of variance.

H—Sequential multivariate techniques including correlation, regression, and stochastic approximation.

I—Stochastic processes. This category includes sequential procedures for Poisson processes, queuing problems and other discrete and continuous stochastic processes per se. Papers in which stochastic process techniques are used to develop sequential analysis theory are included in category *B*.

J—Sequential non-parametric techniques.

K—Sequential procedures dealing with proportions when the underlying distribution is normal. This includes probit analysis and the use of variables for acceptance sampling for proportion defectives.

L—Sequential procedures dealing with non-normal continuous distributions.

Y—Miscellaneous. This category includes other sequential procedures not covered by the preceding categories.

Z—Unclassified. This category is made up of publications which have not been seen by the author and for which no abstract has been found which gives a clear indication of its content.

The numbers in brackets which sometimes follow these letters refer to the year that this particular publication was abstracted in *Mathematical Reviews* and the page on which the abstract may be found. Articles are assumed to be in the same language as the title unless otherwise indicated. Papers marked with an (*) also appear in *Selected Papers in Statistics and Probability by Abraham Wald*, McGraw-Hill Book Company, New York 1955.

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PUBLICATIONS BY ANNOTATION LETTER

A. General and Expository (55)

Anscombe (1953); Armitage (1954, 1957); Arnáiz (1956); Bowker and Goode (1952); Bross (1952, 1953); Burington and May (1953); Burr (1953); Davies (1956); Deming (1950); Eisenhart, Hastay, and Wallis (1947); Ferber (1949); Fiske and Jones (1954); Hald (1952); Harrison and Elder (1950); Hoel (1954a); Johnson, P. (1949); Kenney and Keeping (1951); Lombardi (1951); Mood (1950); Morgenstern (1951); Neyman (1952);

Olds (1950); Ríos (1952); Rosander (1951); Schmetterer (1956); Statistical Research Group, Columbia University (1945); Storch (1953); Wald (1945a, 1945c, 1947a, 1947d); Wolfowitz (1952a); Wünsche (1953).

B. Theory (131)

Alvezian (1959); Albert (1947, 1948, 1954); Anscombe (1949b, 1952, 1953, 1954, 1957); Armitage (1950); Arrow, Blackwell, and Girshick (1949); Bahadur (1954, 1958); Barankin (1949); Barnard (1946, 1949, 1952, 1953, 1954); Bartlett (1946, 1955); Bellman (1957); Bhate (1959); Blackwell (1946, 1947, 1951); Blackwell and Girshick (1946, 1947, 1954); Blasbalg (1957a); Blom (1949); Blyth (1951); Castoldi (1956); Chernoff (1956, 1959); Cox (1952a, 1952b, 1952c); Doob (1953); Dresselaers and Gillis (1951); Dvoretzky, Wald, and Wolfowitz (1950, 1951); Eisenhart, Hastay, and Wallis (1947); Feller (1957); Fraser (1951); Freeman (1944); Gani (1956); Gart (1959); Girshick (1946a, 1946b, 1958); Girshick and Savage (1950); Grundy, Healy, and Rees (1956); Hall (1959); Harris (194[illegible]); Hodges and Lehmann (1950); Hoeffding (1953); Johnson, N. (1957, 1959); Kac (1945); Kemp (1958); Kemperman (1950b); Kenney and Keeping (1951); Kiefer (1957a, 1957b); Kiefer and Weiss (1957); Kolmogorov and Prohorov (1949); Konijn (1953, 1954); Lasman and Williams (1958); LeCam (1954); Lehman (1950, 1955); Lehman and Stein (1950); Leimbacher (1953); Lombardi (1951); Mallows (1953); Matusita (1954); Medhi (1956); Miyasawa (1950); Mood (1950); Moriguti (1956); Nandi (1948); Page (1954a, 1954b); Paulson (1947); Pólya (1946); Rao, C. (1950, 1951); Samuelson (1948); Savage (1954, 1957); Seitz and Winkelbauer (1953); Seth (1949); Silber (1948); Sobel (1953); Špaček (1949); Statistical Research Group, Columbia University (1945); Stein (1946, 1950); Steinhaus (1951); Taguchi (1957); Tsao (1954); Tweedie (1956, 1957a, 1957b); Ville and Schützenberger (1951); Vogel (1956); Wald (1943, 1944, 1945a, 1945b, 1945c, 1946a, 1946b, 1947b, 1947c, 1947d, 1948, 1949, 1950a, 1950b); Wald and Wolfowitz (1948, 1949, 1950); Weiss (1953, 1954, 1956); Winkelbauer (1953); Wolfowitz (1947b, 1951).

C. Binomial Distribution (135)

Albrecht, Maclean, Gulde, and Thompson (1955); Anscombe (1946, 1949a, 1949b, 1954); Anscombe, Godwin, and Plackett (1947); Anscombe and Page (1954); Armitage (1950, 1954, 1957, 1958); Arnáiz (1956); Arrow, Blackwell, and Girshick (1949); Barnard (1944, 1945, 1946); Bartky (1943); Bellman (1956); Bennett (1957); Bhate (1959); Billewicz (1956); Birnbaum (1953); Blackwell (1947); Blackwell and Girshick (1954); Blasbalg (1957b); Bowker and Lieberman (1955); Bradley (1953); Bradt, Johnson, and Karlin (1956); Bradt and Karlin (1956); Breakwell (1956); Bross (1952); Burington and May (1953); Burman (1946); Burr (1953, 1957); Butler (1954); Champernowne (1953); Chernoff (1959); Chung (1950); Cowden (1946, 1957); Cox (1952b, 1952c); Darwin (1959); Davies (1956, 1957); deBoer (1953); DeGroot (1959); Deming (1950); Dixon and Massey (1957); Dodge and Romig (1944); Dumas (1953, 1954); Duncan (1952); Eisenhart, Hastay, and Wallis (1947); Fabian (1956); Feller (1957); Ferber (1949); Finney (1949); Fiske and Jones (1954); Freeman (1944); Freeman, Friedman, Mosteller, and Wallis (1948); Gani, (1956); Gart (1958); Girshick (1946a, 1946b); Girshick, Mosteller, and Savage (1946); Goulden (1952); Hald (1952); Haldane (1945); Hamaker (1949, 1953a, 1953b); Hoel (1954a); Horsnell (1957); Huston, Arnold, Heizer, and Bayley (1953); Isbell (1959); Jones (1952); Kayser (1956); Kempthorne (1957); Kenney and Keeping (1951); Kimball (1950); Lasman and Williams (1958); Lehmann and Scheffé (1950); Lehmann and Stein (1950); Lindley (1957); Lombardi (1951); McCarthy (1947); Méric (1953, 1954a, 1954b, 1955a, 1955b, 1957, 1958); Mood (1950); Moran (1953); Moshman (1949); Narayama (1953); Oakland (1950); Olds (1950); Page (1954a); Paradine and Rivett (1953); Pearch (1947); Plackett (1944, 1948); Pólya (1948); Procassini and Clark (1957); Radkins (1958); Rao, M. (1958); Ríos (1952); Robbins (1956); Robinson (1948); Rosander (1951); Sandelius (1951a); Savage (1947); Schmid (1952); Snell and Armitage (1957); Statistical Research Group, Columbia University (1945); Stockman (1944); Stockman and Armitage (1946); Tweedie (1945, 1952); Vance (1950); vanEeden (1955); Villars (1951); Wagner (1953); Wald (1945a, 1945c, 1947d); Walker, A. (1950); Waters (1955); Wolfowitz (1946, 1947a); Wurtele (1955).

D. Other Discrete Distributions (25)

Anscombe (1949b); Arrow, Blackwell, and Girshick (1949); Birnbaum (1953); Blackwell (1947); Blackwell and Girshick (1954); Burr (1953); Chapman (1952); Cox (1953); Davies (1956); Freeman (1944); Girshick (1946a); Girshick, Rubin, and Sitgreaves (1955); Goulden (1952); Hald (1952); Herbach (1948); Kiefer and Wolfowitz (1956); Lehmann and Stein (1950); McCarthy (1947); Sandelius (1950, 1951b); Statistical Research Group, Columbia University (1945); Wald (1945c); Waters (1955); Wise (1946); Wünsche (1956).

E. Normal Distribution—Central Tendency (64)

Anscombe (1949b, 1952, 1953, 1954); Armitage (1944, 1947, 1954, 1957); Baker (1950); Bangdiwala and Monroe (1958); Barnard (1952); Barraclough and Page (1959); Bhate (1959); Blyth (1951); Burington and May (1953); Burr (1949, 1953); Cox (1952a, 1952c); David and Kruskal (1956); Davies (1956, 1957); DeGroot and Nadler (1958); Dixon and Massey (1957); Eisenhart, Hastay, and Wallis (1947); Ferber (1949); Fiske and Jones (1954); Freeman (1944); Garner (1956); Girshick (1946a); Hald (1952); Hall (1959); Hoel (1954a, 1954b); Kenney and Keeping (1951); Lasman and Williams (1958); Maurice (1957); Mood (1950); National Bureau of Standards (1949, 1951); Page (1954a, 1955); Rao, C. (1950, 1951); Ray (1957b); Resnikoff and Lieberman (1957); Rosander (1951); Rushton (1950, 1952a, 1954); Rushton and Lang (1954); Schwarz (1960); Shainin (1952a, 1952b); Sobel and Wald (1949); Statistical Research Group, Columbia University (1945); Stein (1945); Stein and Wald (1947); Taylor (1956); Wald (1945a, 1945c, 1947d); Waters (1955); Wolfowitz (1950).

F. Normal Distribution—Dispersion (22)

Anscombe (1949b); Bangdiwala and Monroe (1958); Bhate (1959); Birnbaum (1958); Box (1953); Burr (1953); Cox (1949, 1952a); David (1958); Davies (1956); Eisenhart, Hastay, and Wallis (1947); Ferber (1949); Fiske and Jones (1954); Freeman (1944); Girshick (1946a); Hald (1952); Johnson, N. (1954); Nandi (1948); Rao, C. (1950); Rushton (1952b); Statistical Research Group, Columbia University (1945); Wald (1947d).

G. Experimentation and Analysis of Variance (23)

Bangdiwala and Monroe (1958); Bechhofer (1958); Bellman (1956); Bezem (1954); Birnbaum (1958); Box (1954); Box and Wilson (1951); Bradt, Johnson, and Karlin (1956); Bross (1952); Chernoff (1959); Cox (1952a); Eisenhart, Hastay, and Wallis (1947); Fisher (1952); Hoel (1955); Johnson, N. (1953, 1954); Kempthorne, Bancroft, Gowen, and Lush (1954); Kiefer (1953); Ray (1956, 1957a); Robbins (1952); Rushton (1954); Rushton and Lang (1954).

H. Multivariate (37)

Armitage (1950); Bangdiwala and Monroe (1958); Block (1957); Blum (1954a, 1954b, 1958); Burkholder (1956); Chung (1954); Cox (1952a); Derman (1956a, 1956b); Derman and Sacks (1959); Dupač (1957); Dvoretzky (1956); Ghosh (1956); Gulati (1956); Guttman and Guttman (1959); Hodges and Lehmann (1956); Jackson and Bradley (1959); Kallianpur (1954); Kesten (1958); Kiefer (1953); Kiefer and Wolfowitz (1952); Lasman and Williams (1958); Mallows (1953); Nandi (1948); Rao, C. (1947, 1951); Robbins (1952); Robbins and Monro (1951); Sacks (1958); Schmetterer (1953, 1954a, 1954b); Villars (1951); Wolfowitz (1952b, 1956).

I. Stochastic Processes (15)

Anscombe (1953); Anscombe and Page (1954); Bartholomew (1956a, 1956b); Bharucha-Reid (1956, 1958); Birnbaum (1953); Dvoretzky, Kiefer, and Wolfowitz (1953a, 1953b, 1959); Kiefer (1957b); Kiefer and Wolfowitz (1956); Moore (1953); Page (1954a); Wünsche (1956).

J. Non-parametric (6)

Bahadur and Savage (1956); Jírina (1952); Noether (1956); Romaní (1956); Snell and Armitage (1957); Tsao (1954).

K. Proportions and Normal Distribution (7)

Bowker and Goode (1952); Bowker and Lieberman (1955); Breakwell (1956); DeGroot and Nadler (1958); Eisenhart, Hastay, and Wallis (1947); Narayama (1953); Resnikoff and Lieberman (1957).

L. Non-normal Continuous Distributions (14)

Adke and Dharmadhikari (1958); Albert (1956); Anscombe (1952); Box (1953); Cohen (1958); Divatia (1950); Epstein (1955); Epstein and Sobel (1955); Kiefer (1952); Lieberman (1959); Lieberman and Kulin (1957); Sobel (1956a, 1956b); Tomlinson (1957).

Y. Miscellaneous (9)

Bellman (1956; Blackwell and Girshick (1954); Blyth (1951); Bradt, Johnson, and Karlin (1956); Goodman (1953); Hoeffding and Wolfowitz (1958); Kiefer (1953); Robbins (1952); Ville and Schützenberger (1951).

Z. Unclassified (20)

Anastasi (1953); Burgess (1955); Close (1951); Hamaker (1950); Hamaker, Taudin Chabot, and Willemze (1950); Johnson, N. and Moore (1957); Kemperman (1949, 1950a); Lindgren and McElrath (1959); Maguire (1953); Moonan (1950); Noel (1952); Ríos (1954); Rivett (1951); Storch (1953); Vance (1950); Wald (1943); Walker, H. (1949); Wünsche (1952, 1953).

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OCTAVIO CABELLO, 41, received his B.S. in Physics and Mathematics from the Universidad de Chile in 1937 and the degree, Master of Public Health, from the Johns Hopkins School of Public Health in 1945. From 1937–49 he worked in the Department of Biostatistics of the Chilean Public Health Service, also serving as Assistant Professor of Biostatistics, Universidad de Chile, from 1946–9. Since 1949 he has been a Statistician with United Nations. He has published several articles in medical and statistical journals in South America and the United States.

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MANNING FEINLEIB, 25, is a third year medical student at the State University of New York College of Medicine at New York City. He has also taught at this school and has worked for the Statistical Studies section of the Metropolitan Life Insurance Company. Before entering medical school Feinleib majored in Mathematics at Cornell University. The study reported in this issue began as "an almost routine tabulation but developed into an exploration of the still sparsely explored inter-region between theoretical statistics, applied statistics, and the epidemiology of cancer."

HAROLD WOODSON GUTHRIE, 40, has been Assistant Professor at the Cowles Foundation for Research in Economics, Yale University, since 1955 when he received his Ph.D. in Economics from the University of Michigan. From 1947–54 he was a Study Director at the Survey Research Center, University of Michigan. In 1958–9 Guthrie

served as Visiting Professor, Gadjah Mada University, Jogjakarta, Indonesia. Previous articles have appeared in the *Michigan Business Review* and the *American Economic Review*.

JAMES EDWARD JACKSON, 35, studied Economics at the University of Rochester and the University of North Carolina before taking his Ph.D. in Statistics at Virginia Polytechnic Institute. He has been with Eastman Kodak Company, where he is now Senior Analyst, Management Systems Development Department, since 1948 except for periods of graduate study and a brief period as Process Engineer with the Hercules Powder Company. He collaborated with R. A. Bradley on a paper, "Multivariate sequential procedures for testing means," published in *Technical Report No. 10* of the Virginia Agricultural Experiment Station, Blacksburg, 1959, and has had articles in *JASA*, *Technometrics*, *Biometrics* and the *Journal of the Optical Society of America*. The bibliography presented on p. 561 was started "in order to carry on an investigation of some sequential multivariate procedures for testing means."

BRADFORD FISHER KIMBALL, 64, is Director of Public Service Research, New York Public Service Commission. He has been with New York public agencies since 1933 and in his present position since 1956. Earlier he taught Mathematics after receiving his M.A. from Harvard (1921) and his Ph.D. from Cornell (1926) in that field. His many articles in mathematical, statistical and economic journals include papers on pure mathematics, theory of maximum values, use of orthogonal polynomials and applications of mortality models. Kimball's interest in problems of plotting conventions arose from dissatisfaction with some aspects of procedures used to study maximum annual flood flows.

RUPERT GRIEL MILLER, JR., 27, is Assistant Professor of Statistics, Stanford University. His previous articles include: "Contribution to the theory of bulk Queues," *Journal Royal Statistical Society B*, Vol. 21, 1959, pp. 320–37; "Priority queues," *Annals of Mathematical Statistics*, Vol. 31, 1960, pp. 86–103; and "Optimal persistence policies," *Journal Operations Research Society America*, Vol. 8, 1960. He holds an A.B. in Mathematics from Princeton and a Ph.D. in Statistics from Stanford.

JAMES PACHARES, 37, has been with Hughes Aircraft Company since 1954 and has served as Assistant Head, Scientific Computing Services Section, Engineering Division since 1958. He was formerly Consulting Statistician at the Naval Air Missile Test Center Point Mugu, California. Pachares was granted the Ph.D. in Mathematical Statistics from the University of North Carolina in 1953 after he had majored in Mathematics at the University of Akron. Previous articles have appeared in *Annals of Mathematical Statistics*, *IRE Transactions on Engineering Management*, *IRE Transactions on Information Theory* and *Biometrika*. He reports that the need for the tables contained in the article on p. 521 arose in connection with a vendor rating program used at Hughes Aircraft Company.

SIDNEY SIEGEL, 44, studied Statistics and Psychology at Stanford University obtaining his Ph.D. in the latter field in 1953. He served as Instructor and Research Associate at Stanford before going to Pennsylvania State University as Assistant Professor of Statistics and Social Psychology in 1954. He was appointed Professor of Psychology in 1959. In 1957–8, Siegel was a Fellow at the Ford Foundation's Center for Advanced Study in the Behavioral Sciences where the procedure reported in the present article was developed in informal discussions with John Tukey. Siegel has written *Nonparametric Statistics for the Behavioral Sciences*, New York: McGraw-Hill, 1956, and collaborated with D. Davidson and P. Suppes on *Decision Making: An Experimental Approach*, Stanford, California: Stanford University Press, 1957, and with L. E. Fouraker on *Bargaining and Group Decision Making: Experiments in Bilateral Monopoly*, New York: McGraw-Hill, 1960. The last named was awarded the $1,000 monograph prize in the social sciences for 1959 by the American Academy of Arts and Sciences.

JOHN WILDER TUKEY, 44, is Professor of Mathematics, Princeton University. His recent discussion of current problems, "Where do we go from here?," appeared in the *Journal* for March, 1960. Biographical notes are given on p. 215 of that issue.

BOOK REVIEWS

A Time Series Analysis of Interindustry Demands. *Kenneth J. Arrow* and *Marvin Hoffenberg* with the assistance of *Harry Markowitz* and *Ronald Shephard*. Amsterdam, Holland: North-Holland Publishing Co., 1959. Pp. vii, 289. $7.25.

L. R. KLEIN, *University of Pennsylvania*

INPUT-OUTPUT analysis is an accepted branch of econometrics based on a mathematical model and using actual statistical data of economy. Yet it has developed, unlike other branches of econometrics, without stochastic foundations. In a sense, past applications of this technique have been based on *no degrees of freedom* making it impossible to assess the reliability of estimates or make other statistical inferences about the universe. A major contribution of the Arrow-Hoffenberg study is to provide a probability model for input-output analysis. They make actual applications of their model to a sample of industrial data.

The need for a probability model underlying the method is emphasized by its lack of accuracy in prediction. Tests of accuracy in the prediction of industry outputs from a given bill of final demand or, what is the same thing, tests of constancy of the input-output coefficients determined in the usual way from a single period's observations have been made by proponents and critics of input-output. Arrow and Hoffenberg give a useful summary of these tests and show clearly that the coefficients have not remained constant. Regression of individual output on total output and final demand blowups tend to give superior projections.

The facts that input-output projections are decidedly not accurate and that they do not appear to deviate randomly from the observed values lead the authors to propose some new structural relationships in which residuals (between projected and observed outputs) are related to a selected number of important economic variables. They choose real per capita disposable income, the ratio of defense expenditure on goods to private GNP, a time trend, and the ratio of positive deviations of output from past peak levels to current output.

Their model takes the form[1]

$$a_{ij}(t) = b_{ij} + \sum_{k=1}^{4} c_{ijk} y_k(t)$$

$$x_i(t) = \sum_{j=1}^{n} b_{ij} x_j(t) + \sum_{j=1}^{n} \sum_{k=1}^{4} c_{ijk} x_j(t) y_k(t) + f_i(t) + u_i(t)$$

$a_{ij}(t)$ = variable input-output coefficient of period t

$y_k(t)$ = variable explaining fluctuations in $a_{ij}(t)$

$x_i(t)$ = output of sector i in period t

$f_i(t)$ = final demand for sector i in period t

$u_i(t)$ = random disturbance.

The b_{ij} and c_{ijk} are unknown structural parameters. Essentially, the Arrow-Hoffenberg method uses time series observations on $x_i(t)$, $y_k(t)$, and $f_i(t)$ to estimate b_{ij} and c_{ijk} from the second of the two equations or a transformation of it. For individual years, there are not enough data available to make direct estimates of the first equations. It would seem appropriate to allow for stochastic variation in the first as well as in the second of the two above equations. This would have made some difference in the statistical treatment.[2]

[1] A similar treatment is given by F. E. A. Briggs, "On problems of estimation in Leontief models," *Econometrica*, Vol. 25, July, 1957, pp. 444–55.

[2] *Ibid.*

The four particular variables mentioned above, $y_k(t)$ $(k=1,2,3,4)$ were used as factors of major practical importance. Relative prices were not included because it was felt that production is inelastic in the short run, but, surely, the input-output scheme is a long run model used at its best for the analysis of long run problems.

The book contains an excellent discussion of problems of statistical inference for the type of model considered. There is a good description of methods of estimation in systems of simultaneous linear equations. For computational ease, they decide upon the use of Theil's k-class of estimators with a particular value assigned to k. In view of present computing technology, there is little reason to shun limited information maximum likelihood because of cost considerations. The work breaks new ground with development of linear programming estimates, which take account of the non-negativity and other limitations on the size of input-output coefficients. In the end, better results are obtained for the linear programming than for the ordinary equation systems method of estimation.

The authors use, as a point of departure, the 1947 interindustry study. Their time series values cover the years 1929–50. They consolidate the 1947 study into 66 sectors. It is a "truncated" model in that many industry sectors are included in final demand. They carefully consider the major and negligible flows in their layout. Ultimately they estimate equations for variable input-output coefficients $a_{ij}(t)$, in only four sectors. They are Lumber and Wood Products, Crude Petroleum, Petroleum Products, and Rubber Products. In each of the four cases, they consider the $a_{ij}(t)$ to be variable for only a limited number of consuming sectors, at most eight and do, not use all four explanatory variables simultaneously. They had to make use of a number of other predetermined variables that would be relevant to a more complete model but that did not appear in the structural equations being estimated. With these, they were plagued with problems of multicollinearity and shortage of degrees of freedom.

In general they come up with useful relationships explaining fluctuations in those $a_{ij}(t)$ that they chose to investigate. Their approach is highly interesting, ingenious, and properly executed. They frankly admit, however, that their findings lack positive power. They have not found a set of numerical values that are going to give genuine improvements of large magnitude over conventional input-output results using the 1947 tables. They are perhaps too sweeping in their pessimism concerning the general futility of time series analysis. They say, "The complexity of the model makes it difficult to see clearly where it is wrong." This reviewer would say that their brilliant new tools should have been applied more cautiously in a smaller, more aggregative, self-contained model with a gradual build-up of success upon success to the finer classifications that they chose.

* **Elementary Decision Theory.** *Herman Chernoff* and *Lincoln E. Moses.* New York: John Wiley and Sons, Inc. London: Chapman and Hall, Ltd., 1959. Pp. xv, 364. $7.50.

LIONEL WEISS, *Cornell University*

SINCE this text represents a departure from the topics usually covered at this mathematical level, it seems desirable to start this review with a list of the main topics covered, chapter by chapter.

Chapter 1: Decision making when there is uncertainty about the state of nature; strategies; average losses; admissibility and inadmissibility of a strategy; minimax average loss strategies; minimax average regret strategies.

Chapter 2: Frequency distributions; histograms; cumulative frequency polygons; summation signs; sample mean and variance and their computation.

Chapter 3: Probability as a long-run frequency; probability distributions and cu-

mulative distribution functions; random samples; the normal distribution; sets and functions and their use in probability theory.

Chapter 4: Utility as a measure of value when the probability distribution is known; probability and expectation; application of utility to fair bets; the St. Petersburg Paradox; parameters of probability distributions.

Chapter 5: Uncertainty due to the ignorance of the state of nature; randomized strategies; convex sets; Bayes strategies and supporting planes; minimax strategies; regret.

Chapter 6: A posteriori probability; conditional probability; computation of Bayes strategies; independence.

Chapter 7: Introduction to classical statistics; examples of hypothesis testing and of point and interval estimation; power; principles for setting the level of significance.

Chapter 8: Mathematical models.

Chapter 9: A summary of terminology and methods of testing hypotheses.

Chapter 10: A summary of terminology and methods for point and interval estimation.

Tables, and appendixes on notation, shortcut computational formulas, axioms of probability, properties of expectation, likelihood ratio tests, game theory, utility functions.

In their preface, the authors recommend the first seven chapters for a first course in the fundamental ideas of statistics, requiring only high school mathematics. They feel that some background in statistics or mathematics would be desirable before attempting to cover chapters 9 and 10.

The text is very clearly written, and there are many illustrative examples and exercises. The reviewer feels that the arrangement of the material should have been made somewhat tighter: for example, the sections on probability theory are scattered in various parts of the text. Many of the appendixes are on a higher mathematical level than the main part of the text, and it is doubtful whether a student who would be challenged by the first seven chapters would be able to handle these appendixes.

Although there is little doubt that as time goes on more and more first courses in statistics will be given from the decision theory point of view, at present this is rarely done. One reason for this inertia has been the lack of suitable textbooks. The text by Chernoff and Moses requires more mathematical maturity (which is not identical with formal course work) than most of the current pre-calculus texts for first courses in statistics, but it should be considered by those instructors who would like to introduce a more modern (and a more logical) approach in a basic course in statistics.

Operations Research: An Annotated Bibliography. *James H. Batchelor.* Saint Louis: Saint Louis University Press, 1959. Pp. ix, 866.

O. H. BROWNLEE, *University of Minnesota*

IN THIS volume are listed and (in many cases) abstracted more than 4,000 books and papers having some relevance to that nebulously defined area "operations research." The papers listed include *all* of those from some American, British, Italian, French, German, and Japanese journals devoted to operations research or one or more of its fields as well as reports and published materials from conferences and meetings. There is an 84-page index classifying the listed materials (by number) according to author and subject matter. The bibliography extends a similar one, *Operations Research: A Preliminary Annotated Bibliography*, published in 1952, to the end of 1957.

A useful appraisal of any classification of knowledge requires familiarity with the subject matter, the available materials, and the demands and competencies of the

persons for whom the classification is prepared. The subject matter of operations research is by no means clearly defined. In fact there is a large volume of literature addressed to the question, "what is operations research?" The field appears to be defined both with reference to kinds of problems and techniques for solution. In this reviewer's judgment, the author certainly has not interpreted the field too narrowly.

Because of the broad definition of the field, materials relating to such methods of formulating and solving problems as various forms of programming (linear, nonlinear, parametric, dynamic, etc.), queueing theory, game theory, and simulation as well as to broad classes of knowledge such as statistics, mathematics, economic analysis, accounting, and psychology could be considered as operations research materials. To provide a comprehensive listing of books and papers from all of these areas that have applications to operations research is an accomplishment that cannot reasonably be expected of a single bibliographer. However, a user of the bibliography would hope that really important things are listed in the bibliography and that the work contains a minimum of materials of dubious value.

In the reviewer's opinion, an excessive number of works of dubious value (particularly newspaper reports and items from trade or business journals) have been listed. Perhaps the author wished to compile a bibliography for both businessmen and research workers or practitioners. Also, he states that he prefers to minimize errors of omission by accepting the author's or editor's indication that the work pertains to operations research. However, this procedure apparently is not always sufficient for among the items of importance that are omitted is Feller's *An Introduction to Probability Theory and Its Applications.*

The bibliography should prove useful. Its detailed index and the abstracts are particularly attractive features. It classifies some unimportant materials and omits some important ones; but can there be a bibliography on this subject which doesn't make such mistakes?

Variables Related to Human Breast Cancer. *V. Elving Anderson, Harold O. Goodman* and *Sheldon C. Reed.* Minneapolis: The University of Minnesota Press, 1958. Pp. xiii, 172. $4.00.

HAROLD F. DORN, *National Institutes of Health*

THE development of cancer by two or more closely related members of the same family has attracted the attention of physicians and geneticists for at least two hundred years. This fact was first interpreted by many as demonstrating that cancer is a genetically determined disease. It was gradually recognized, however, that multiple cases in the same family might be merely coincidental rather than an expression of inherited susceptibility and that different types of cancer probably are separate diseases with different specific etiological agents. Furthermore, statistical studies of the incidence of cancer in various populations revealed large differences in morbidity and mortality from cancer associated with demographic, cultural and environmental characteristics, which suggested that both exogenous and endogenous factors influence the development of cancer. Except for a few rare forms of human cancer the disentanglement of the relative importance of extrinsic and intrinsic factors in the development of cancer has proven to be unexpectedly difficult.

The authors of this study sought the answer to the question, "do the relatives of breast cancer patients have an increased risk of developing cancer, and if so, is this an increased risk for cancer in general or for specific forms of cancer?" The book begins with a concise review of the methods of study used by previous investigators and of the problems encountered in the interpretation of the results of these studies.

After reviewing the advantages and disadvantages of the three principal methods—the study of family pedigrees, the analysis of the history of twins, and the analysis of data about relatives of persons with cancer—the authors selected the latter method as the most suitable for their purpose.

The remainder of the book is devoted to a thorough discussion of the four main problems that are encountered in studies of this kind, (a) how shall the index cases or propositae be selected, (b) what control or comparison group shall be used, (c) how can the data be validated, (d) what are the most appropriate methods of analysis. Each problem is discussed in meticulous detail, procedures taken to solve the problems are clearly set forth, and the results are frankly evaluated.

The data are from the histories of the families and relatives of 544 women who were examined at the Tumor Clinic of the University of Minnesota Hospitals. The control group was the parents and sibs of the husbands of these women. As a check upon the conclusions drawn from the analysis of the histories of these persons, a second set of propositae was chosen from the records of the University of Minnesota hospitals for the years 1910–1925 and from the University of Minnesota Pathology Department records for the years 1910–1920.

The data for each of the separate groups studied are carefully described and thoroughly analyzed for possible biases that might affect the interpretation of the data. The variation in the conclusions that would be drawn by using different control groups is illustrated. A variety of methods of adjusting and analyzing the data are tried and the results are contrasted. Throughout, special attention is paid to the validity of information used. This book should be a required study for all investigators planning to undertake a retrospective population study of the etiology of disease. In its detailed description of methodology, thorough search for possible biases in the data that might affect the interpretation of the results, the examination of the effect of choosing different control or comparison groups, the exposition of the different conclusions reached by alternative methods of analysis, and the competent and critical handling of the data, this study sets a high standard for future investigators.

State Income Differentials, 1919–1954. *Frank A. Hanna.* Durham: Duke University Press, 1959. Pp. xix, 268. $7.50.

ERNEST W. GROVE, *U. S. Department of Agriculture*

THE final report of the Study of Differences in State Per Capita Incomes, financed jointly by Duke University and the Rockefeller Foundation, this volume is an intensive survey and detailed statistical analysis of the income data available for States. Differences among States, and changes in those differences, are studied in terms of the official Department of Commerce estimates of State personal income back to 1929 and the 1919–21 estimates prepared by Maurice Levin for the National Bureau of Economic Research.

Time series analysis, attempting to separate cyclical and secular from random elements in income differentials, is supplemented by cross-section analysis based on 1949 income data obtained in the 1950 Census of Population and earnings data for 1947 and 1948 from the Censuses of Manufactures and Business. The cross-section analyses concentrate on "explanatory" variables associated with State differences in per capita incomes, principally occupation and industry.

Much of the material in this volume has appeared in earlier publications, notably in the volume on "Regional Income"[1] reporting the proceedings of the June 1955 meeting of the Conference on Research in Income and Wealth. There is some new

[1] Volume 21 of Studies in Income and Wealth, National Bureau of Economic Research, Princeton University Press, 1957.

material, however, and it was worth while to bring it all together in this final report comprising 7 chapters and 3 appendices.

Chapter 1 is an excellent summary, not too brief, for the benefit of readers who do not wish to follow all of the detailed presentations in subsequent chapters. Chapter 2 analyzes the "Patterns of Differential Change" in State per captia incomes. The rank order of States has not changed very much since 1919, but there has been a pronounced narrowing of relative interstate differentials as measured by coefficients of variation. These coefficients tend to move contracyclically. In other words, as national income per capita increases, its relative interstate dispersion tends to decrease, and vice versa. Professor Hanna calls this the "accordion effect."

Some regression of the State per capita incomes toward their mean as income increases is indicated. Hanna proceeds to the measurement of these regression lines in very great detail. Extensive work based on the old State income payment series developed regression analyses of the changes in State per capita incomes between all possible pairs of years from 1929 to 1949, and between each of the years 1919–21 paired with each other and with 1929, 1933, 1939, and 1949. This formidable statistical undertaking was supplemented by some analyses along the same line with the newer personal income data for 1929–54. These intertemporal regressions are the weakest element in the whole volume, for their results are not only inconclusive (e.g., as to linearity), but also lacking in any real significance. Some experimentation and summary presentation of the results would have been justified, but the details in Appendix A on "Regression and Differential Change" might well have been omitted.[2]

Chapter 3, "Cyclical Behavior of Differentials," develops "sensitivity" and "time-factor" indexes for each State, designed to measure cyclical and secular changes, respectively. Because farm income is irregular in its behavior—actually negative in some States for some years in the 1930's—the indexes were computed separately for farm and nonfarm income as well as the total. The results indicate that lower income States are generally more sensitive to cyclical changes than higher income States. They also have more and bigger upward trends, as indicated by the time-factor indexes.

Chapter 4 subjects the per capita "Income Components"—labor income, property income, transfer payments, farm proprietors' income, and nonfarm proprietors' income—to the same type of analysis. The "accordion effect" is found for all components except farm proprietors' income, which was also the only component to show an increase in relative interstate dispersion from 1929–54. It is concluded that the downward trend in the relative interstate dispersion of total personal income per capita is due in large measure to downward trends in the relative size and interstate dispersion of property income. Appendix C contains the detailed State estimates of the per capita income components.

Chapters 5 and 6 deal with "Occupational Earnings" and "Industrial Earnings," respectively. In each case, a "rate-constant earning" is computed for each State, the national average occupational or industrial earnings rates being applied to the occupational or industrial distribution of the individual State. In other words, each State is "standardized" for earnings rates, yielding an average rate whose deviation from the national average is a reflection of the State's deviation from the national average industrial or occupational composition. In both cases, rate-constant earnings above the national average are mostly in the Middle Atlantic and East North Central regions, whereas those below the national average are mostly in the South.

There is a tendency for States with unfavorable occupational or industrial compositions to have earnings rates below the national averages for their industries and occu-

[2] As Edward F. Denison suggested 5 years ago (*ibid.*, p. 175).

pations; and conversely, there is a tendency for States with favorable compositions to have earnings rates above the national averages. This correlation, unfortunately, makes impossible any exact assessment of the relative importance of rates and composition in accounting for interstate differences. Hanna does the best he can in the face of this difficulty, and devotes Appendix B to a general discussion of the problem of standardization.

In Chapter 7, "Demographic Factors," several population variants, all more closely associated with income production than is total population, were substituted as denominators in computing per capita income. States with larger-than-average percentages of their population of working age are in the Northeast or on the Pacific Coast. Those with larger percentages under or over working age are in the Southern and Mountain regions.

This study was needed. It was well conceived, and its execution has been conscientious and painstaking. The statistical techniques employed are unexceptionable, and they have been applied with great diligence. The original claim staked out by Professor Hanna was an important one, and he has worked it thoroughly. Although the ore proved out at a lower grade than was originally expected, much good metal was produced nevertheless.

The results of this study will certainly be of value to State planning and development commissions, and perhaps also to local agencies concerned with urban development. Students of economic growth may find some new grist for their presently fashionable mill, but the results generally seem a little disappointing on this score. Familiarity with the results from previous publications may be partly responsible. But mainly it is due to the relatively short period covered by the study, and to the fact that generally rising incomes in the last 20 years have made it difficult to separate cyclical from secular forces affecting State per capita incomes. Considering the limitations inherent in the study, its results are generally quite reasonable—and not unexpected. But the author cannot be blamed for the absence of any startling conclusions.

Secret Detention by the Chicago Police. *American Civil Liberties Union, Illinois Division,* Glencoe, Illinois: The Free Press, 1959. Pp. 47, $1.00 (paper).

DANIEL GLASER, *University of Illinois*

THIS pamphlet reports a tabulation of time between arrest and booking for a sample of cases tried in Chicago's Municipal Court. The law requires that arrestees be brought before a magistrate "without unnecessary delay," but the study found that in the nine branches of the court on which it reports, 25 per cent were held over 17 hours, and 2.5 per cent over 48 hours. In one branch, 50 per cent were held over 17 hours and 9 per cent over 48 hours. This conclusion is based on time of arrest and of booking entered by police on their Arrest Slips, but omits a third of the slips, in which the police left these sections blank. Furthermore, the period between arrest and booking may constitute only the first of several "unnecessary delays" in bringing an arrestee to court.

Sampling for this research was initiated by a lawyer who pulled ten or twenty files from each of several cabinet drawers. Statisticians Kruskal and Meier of the University of Chicago were consulted later. They contributed appendices describing and appraising the sampling procedures, and estimating confidence limits on the assumption that the data approximate a probability sample. Two possible types of bias may be of interest. First, of course, the effort to sample mechanically all parts of the file drawer means that thicker files have more chance of being drawn. Secondly, personal bias of the tabulators for thicker or thinner files may develop, for example, if the thinner files

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yield the documents sought more readily, or if they are more often completed in the thicker files.

This is only the most recent of a long series of such statistical "exposures" of the gap between prescription and practice in American law. The consistency of findings in various jurisdictions suggest that similar illegal practices can be found today in most American cities, and regardless of laws excluding illegally obtained evidence, which this pamphlet recommends. This suggests the futility of "crying with alarm," rather than extending the research to expose correlates of the defects, which might suggest remedies.

That the latter possibility was overlooked in this study is suggested by the last Appendix, which reports some findings on eleven branches of the court in which 352 cases were sampled, but which were excluded in favor of concentrating on 2,038 cases from nine other branches. Less than five per cent of the excluded cases were delayed over 17 hours, and only nine per cent of these cases had no time information. Five of the excluded branches, from which 132 cases were drawn, had no delay over 17 hours. It may well be that a much greater contribution to the reduction of unnecessary delay might have been made through contrast of the branches to which one might "point with pride" with those branches which yielded the findings which the pamphlet deplores. Within the branches studied, a comparison of the delayed with the non-delayed cases would have been of interest.

Social action organizations can contribute to social science by pointing out problems with which the public is concerned. They generate an often necessary heat. But social scientists must restudy the problems, and be less exclusively concerned with raising temperatures, if there is to be light.

A Manual for Economic and Functional Classification of Government Transactions. *United Nations, Department of Economics and Social Affairs.* New York: Columbia University Press, 1958. Pp. x, 188. $2.00.

NANCY D. RUGGLES, *Yale University*

THE United Nations *Manual for Economic and Functional Classification of Government Transactions* is the result of a gradual evolution of work on this subject in the Fiscal and Financial Branch of the United Nations. As early as 1951 the United Nations had published a study entitled *Budgetary Structure and Classification of Government Accounts* (Sales No. 1951.XVI.3). The purpose of the manual under review is to develop classification schemes which will provide meaningful data on government transactions. It is unfortunately true that despite the central role of government in most economies today the traditional forms of public sector accounts do not provide the kinds of information required for making economic and social policy decisions. The traditional appropriation accounts are concerned with providing funds to specific administrative units within the government, and these requests are usually cast in terms of the object of expenditure (such as wages, travel, buildings, etc.) rather than the purpose or function of the expenditures. Furthermore, they are usually prepared at an extremely detailed level, without the sorts of summarization that are useful for policy purposes. On the other hand, the government sector in the national income accounts is also grossly inadequate for policy-making purposes. Although economic classifications are generally used in the national economic accounts, the degree of consolidation is such that only the very broad outlines of government activity are revealed. Furthermore, as in the appropriation accounts, little or no information is provided on the purpose or function of the various kinds of government activities. The United Nations *Manual* attempts to remedy this situation by setting forth a system

of tables into which the basic government accounts can be reclassified. Since the introduction of the *Manual*, a number of countries have undertaken reclassification projects along the lines suggested in the *Manual*.

The *Manual* is primarily concerned with the development of procedures for obtaining an economic classification of government expenditures. For this purpose four types of governmental entities are distinguished. These are (1) government trading enterprises, which produce goods and services for sale to the general public but are obliged to surrender any surplus they may earn to the general government funds; (2) ancillary agencies, which are organized to supply goods and services for use of individual government agencies but which do not provide services to the public; (3) government financial enterprises, which are organized for purchase or sale of financial claims, issuing domestic currency, foreign exchange, etc.; and (4) general government agencies, which are financed primarily by specific appropriations from general government funds. For each of these types of agency, on both the central and the local levels, the *Manual* sets up a system of economic classification of transactions, distinguishing such flows as current expenditures on goods and services, transfers, subsidies, interest on government debt, etc. Considerable attention is given to setting up sets of accounts which are appropriate to the different types of government agencies, and to the principles of consolidating the accounts for individual sectors into a general account for the government as a whole.

Secondarily, the *Manual* is concerned with the principles of functional classification of government expenditures. A specific functional classification scheme is proposed, in which the main categories are general services, community services, social services, economic services, and unallocable expenditures. General services include such things as general administration, defense, justice, and police. Community services cover roads, fire protection, and other community services. Social services cover education, health, social security, housing, and recreation. Economic services cover aids to agriculture, fuel and power, mineral resources, transportation, and similar services. Unallocable expenditures include such things as interest on the debt, inter-governmental transfers, and foreign economic aid. The functional classifications are linked with the economic classifications, providing a cross classification for the government sector as a whole of expenditures by both economic character and function.

Finally, a chapter is devoted to the reconciliation of the government sector accounts as shown in the *Manual* with the government sector in the United Nations system of national income accounts (*A System of National Accounts and Supporting Tables*, UN Sales No. 1952.XVII.4). Considerable difficulty arises in this endeavor, however, inasmuch as the United Nations system of national income accounts was not designed with any thought of integration with any other form of economic accounts. The attempt at integration is, therefore, somewhat strained and complex. The government sector accounts themselves can serve a useful function in providing information for policy makers, but it seems doubtful whether this function would be enhanced by this kind of reconciliation with the national income accounts. There is unquestionably a need for increased attention to the integration of all of the forms of national economic accounting, but the need goes far deeper than the provision of complex reconciliation formulae for basically dissimilar accounting systems. It might therefore be far better not to force the reconciliation of the government sector accounts with the national income accounts at this point, but rather at some juncture to reconsider the redesign of the national income accounts themselves so that they can be logically integrated not only with policy oriented public sector accounts, but with such other forms of economic accounting as input-output and flow of funds accounts.

The Measurement of National Wealth, Income and Wealth Series VIII. *Raymond Goldsmith* and *Christopher Saunders*, Eds. London: Bowes and Bowes, 1959. Pp. xiv, 389. 45s.

MORDECHAI E. KREININ, *Michigan State University*

CLOSELY related to the development of aggregate economic analysis was the introduction and widespread adoption of national income accounting. While these accounts proved extremely useful in filling the data requirements for the Keynesian analytical framework, it is becoming increasingly apparent that they must be supplemented by other sets of accounts such as estimates of national wealth. Not only do these estimates serve as an essential guide for policy measures directed at short-run economic fluctuations; they are also indispensable for a thorough understanding of the long-run process of economic growth.

The volume under review represents a compilation of national wealth data in a variety of countries. The first chapter, a summary survey by Th. D. Van Der Weide, contains statistics on national wealth for eighteen nations from five continents. For each economy it provides the structure of national wealth at a recent time and some capital-output ratios over time. No attempt is made to analyze the data, the task of interpretation being left to chapters 4 through 13,[1] each of which is devoted to an individual country or region. The chapters are written by experts closely associated with the development of the accounts in their respective countries. (Incidentally, it would have been useful to know the professional association of the authors.) In order of appearance, these countries are: Norway, the Netherlands, Germany, Yugoslavia, Canada, Latin America, Argentina, South Africa, Australia and India. Although the amount of data available varies greatly from one country to another, all the chapters have certain elements in common.

First they all explore in detail the methodological problems involved in compiling the estimates, such as sources of data and methods of estimation. There are two basic methods for estimating the stock of capital assets. "The first is the 'perpetual inventory method' consisting of the cumulation of statistics of gross capital expenditures over time, making allowance for depreciation or disappearance of old assets and for 'changes in prices" (p. 4). This procedure requires: a) estimates of capital expenditures over long periods in the past; b) some basis for estimating the length of life of capital goods; and c) price series for capital goods. The second method is a direct estimate of the stock of assets in use at a given time, derived from official censuses, fire insurance valuations, the balance sheets of enterprises, capitalization of income, and tax statistics. The method used in any particular instance depends largely on the types of information available.

Second, all chapters present estimates of aggregate national wealth and its composition. For many of the countries these estimates go back over a protracted period of time. Although the introductory chapter suggests a uniform scheme for presenting the composition of wealth (p. 4),[2] there are great variations in the manner of presentation followed by individual authors. Third, most chapters contain capital-output ratios for the entire economy either over time or at a given time.

While the mere compilation of these statistics is in itself of great value, the general economist would be more interested in interpretation of the data. In that respect the

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[1] Chapters 2 and 3 are devoted to methods of measuring capital and to financial accounting, respectively.

[2]
A. *Enterprises*
  I. Reproducible Assets
    Structures
    Equipment
    Inventories
  II. Non-reproducible assets (Land)
B. *Government*
  I. Reproducible Assets
    Structures
    Equipment
    Inventories
  II. Non-reproducible assets (Land)
C. *Households (durable goods)*
D. *Foreign assets (net)*

chapters on Norway (4) and Latin America (9) are probably the most outstanding.

After presenting the wealth and income estimates for Norway for the first half of the 19th century, the authors (O. Aukrust and J. Bjerke) compute marginal capital-output ratios for seven sub-periods. From an approximate level of 3 during 1900–1939, the capital-output ratio dropped in World War II, but rose sharply after 1948 reaching 5.5 in 1951–56. These movements are readily explained in terms of shifts in employment and changes in the marginal productivity of capital. A Cobb-Douglas production function is subsequently fitted to the data for 1900–1929 and 1946–1955 with the following results:

$$y_t = 2.262 \cdot C_t^{0.203} \cdot L_t^{0.763} \cdot e^{0.01814}$$

"(1) A partial increase in the volume of real capital by 1 per cent will, *ceteris paribus*, raise the national product by 0.2 per cent. (2) A partial increase of labor by 1 per cent will raise the national product by 0.76 per cent. (3) With constant capital volume and constant employment the national product will, as a result of gradually improving 'techniques,' increase at the rate of approximately 1.8 per cent per annum" (p. 106).

Although this formulation is subject to some important limitations (discussed on p. 109), several inferences can nonetheless be drawn from it. The role of capital in increasing production appears to be considerably smaller than is generally assumed. "The rate of growth of the national product is affected comparatively little by the level of investment. . . . Whether the rate of investment within reasonable limits is high or low, the national product with constant employment will rise by 2–3 per cent per annum, largely because the technical factor alone automatically warrants a growth which has been estimated roughly 1.8 per cent[3] per annum." One implication of this finding is that in an economy like Norway, it is not necessarily desirable to pursue a stringent policy designed to stimulate high level of investments. Better results may perhaps be achieved by diverting resources from investments in reproducible capital to investments in human resources such as education and research.

In the case of Latin America (chapter written by A. Ganz), the analysis is focused on the behavior of the average capital-output ratio over time. The decline of that ratio between 1925 and 1948 reflected a continuous increase in the productivity of capital associated with the process of development, industrialization and better utilization of industrial capacity. Since 1948, however, the ratio remained about constant. Factors such as the inefficient use of land resources and misutilization of public investment resources (p. 234) were found responsible for halting the increased productivity of capital. Following the general analysis of Latin America, economic development in Argentina and Colombia are contrasted and explained in terms of the composition of investments and capital.

Other chapters are also not lacking in analytical insights. In West Germany (Chapter written by F. Grünig) the decline in the capital-output ratio during the postwar period was a reflection of the increased importance of manufacturing as compared with transport and housing and of the increasing rationalization of the production process. The studies of South Africa (authored by D. Franzsen and J. Willers) and Australia (by J. Garland and R. Goldsmith) use the accumulation of national wealth as a focal point in explaining economic development during the past half a century. As may be expected, a major role in the development process is assigned to changes in the terms of trade.

On the basis of the data gathered thus far Mr. Ivo Vinski, author of the chapter on Yugoslavia, suggests the following tentative conclusion about the order of magnitude of the capital-income ratio in various stages of economic growth (p. 165):

[3] Note that one limitation of the analysis is that it assumes the "technique" factor to be independent of the capital factor.

"1) In under-developed countries this ratio seems to be low, probably under 2.5, because labor is the main factor of production in these countries. . . . 2) In countries undergoing the process of accelerated basic industrialization the ratio may rise to over 4.0. . . . 3) In what are now advanced countries the ratio generally ranges between 2.5 and 3.5, the decline being caused by a more economical utilization of capital, increase of productivity, inventions, etc."

The foregoing paragraphs illustrate some uses to which estimates of national wealth can be put, especially when they correspond well to national income data. Attempts to explain the historical behavior of measures such as the capital-output ratio may lead not only to better understanding of the development process, but also to important policy conclusions in the fields of development and resource allocation. The volume under review is therefore a welcome addition to the growing literature on the subject.

Studies in Company Finance, A Symposium on the Economic Analysis and Interpretation of British Company Accounts. Eds., *Brian Tew* and *R. F. Henderson*. Cambridge, England: Cambridge University Press, 1959. Pp. xix, 301. $6.50.

EDWIN KUH, *Massachusetts Institute of Technology*

THIS book is a collection of essays analyzing balance sheets and balance sheet data for about 2500 to 3000 firms whose securities are traded on the British Stock Exchange and whose products fall within the broad categories of manufacturing, building, or distribution for the years 1949–53. The essays have one unifying theme—all are analyses of a homogeneous body of data, either sources and uses of funds statements or balance sheets, which have been put into standardized accounting framework so that frequency distributions, principally in the form of ratios, will not be distorted by capricious inter-firm accounting differences. This approach represents an interesting and potentially extremely fruitful collaboration between the British Government and a private research organization which might well be emulated in this country.

A central tenet is well expressed by the editors in the introductory essay: "We have looked upon ourselves as pioneers exploiting a new and rich mine of economic data, and like pioneers have not hesitated to exploit first the seam which appeared easiest to work." Before pointing to some limitations of this particular approach, it seems best to indicate some of the benefits by reporting the main findings, many of which are of very real interest.

In a discussion of "Dividend Policy and Income Appropriation," S. J. Prais shows that for all firms in the United Kingdom, sharp tax increases in the postwar period fell upon dividends instead of retained earnings. Dividends fell from about 70% of income in 1938 to 30–35% in 1948–56. Taxation rose from about 15% to 40% of net income before taxes over this period while "net savings" rose from 18% to an average of about 25%. Frequency distributions of retained earnings to net income show that 70% of the companies saved between 40% and 80% of their net income. When the same saving-income ratio is distributed according to the rate of growth, it is found that the faster a firm's rate of growth, the higher will be its average saving-income ratio.

In an article of major interest to economists, Brian Tew discusses corporate self-financing. Net investment is defined as the increase in net total assets over the period 1949–53, and corporate saving is defined as net retained earnings, i.e., net earnings less dividends. While some small attention is paid to the role of current depreciation expense, it is largely in another connection. In my opinion, this is unfortunate because the subsequent findings are heavily qualified by this conceptual problem. The ratio of saving to investment is positively correlated with the savings-income ratio; the ratio of savings to investment is inversely (and strongly) correlated with the ratio of income

to investment. Most interesting, they find that the correlation between the log of the savings-income ratio and the log of the investment-income ratio is "remarkably low, only 0.15. . . ." It appears, then, that the cross-section correlation between savings and investment, both divided by corporate income, is very small. Tew also considers other internal sources of finance including the change in net liquid position which unfortunately is defined as liquid assets, i.e., cash, plus tax reserve certificates, plus marketable securities, minus bank credit outstanding and current account accrual such as for dividends, interest, and taxes. Repeated use of net accounts, when observed data may often be attributable to diverse underlying behavior appears to be a shortcoming common to much of the analysis, although clearly opinions on this score can differ quite legitimately.

R. F. Henderson develops evidence on a clearly defined relation between the rate of growth of net assets and new issues. He also reports for the aggregates that the large companies financed a larger proportion of growth by new issues than did small companies and placed particular reliance upon common stock issues. Bank credit was of trivial importance in financing the growth of British firms as a whole, although it was of real importance to certain classes of companies. During this period, small firms did not rely disproportionately on net trade credit (i.e., accounts receivable less accounts payable), although the fastest growing companies found trade credit to be a use rather than a source of funds. Trade credit is primarily used in a fairly rigid relation to output and is seldom an important fund source for firms who are short of funds, although a severe depression could easily alter the observed pattern.

Prais in another essay shows that the hundred largest companies increased their share of gross trading profit (net income plus depreciation) from about 25.2% in 1949 to 31.5% in 1955. In every year large company profits either increased more rapidly or declined less than for all other companies. Studies using these data for individual industries (brewing, cotton cloth, construction, construction materials, electrical goods and machinery, and retail distribution) are also included.

The advantages to different analyses of the same body of data are considerable and hardly require further elaboration. The disadvantages (and these for the most part have been recognized by the authors) should not be overlooked. First, by ignoring supplementary information available in greater detail for a sub-set of firms, the investigators perforce sacrifice a good deal of information in pursuit of homogeneity. Second, the economies of scale and improvements in quality which accrue when a common body of data has been processed according to standard procedures is to some extent offset by the fact that the motivation of the investigation is imprecise.

This study represents a desirable tendency to make available homogeneous and relevant data where none existed before. Over a number of years this approach will yield great benefits since later investigators with more sharply defined objectives will not be strapped by paucity of data. This indeed represents a major advance, particularly in the age of the high speed electronic computer, when data processing is no longer prohibitively expensive.

A General Theory of the Price Level, Output, Income Distribution, and Economic Growth. *Sidney Weintraub.* Philadelphia: Chilton Company, 1959. Pp. 123. $3.50.

HANS BREMS, *University of Illinois*

IF N = employment, P = the price level, Q = physical output, and w = the money wage, Weintraub finds that the ratio PQ/wN has remained relatively constant during the last thirty years. The ratio is called k, and its constancy is called "the law of k." The demonstration of the constancy of k occupies the first half of the book. The demonstration uses extremely primitive tools, i.e., time series for k. The burden of the proof

is carried by two diagrams showing k itself as well as an index of k as functions of time. Why both? In effect, looking for constancy of the average k, rather than the marginal k, implies that a straight line through zero in a PQ-wN diagram represents the best fit. Why not a simple regression line with a nonzero intercept? It would have been less restrictive and might have given a better fit. But Weintraub uses no regressions at all.

The relative constancy of k has been observed by many writers, and it has been interpreted in terms of economic behavior before. In Lawrence R. Klein, *A Textbook of Econometrics*, pp. 75–6, for example, one finds a lucid demonstration that if the production function is of the Cobb-Douglas type, if markets are purely competitive, and if entrepreneurs maximize their profits, then k will be equal to the reciprocal of the elasticity of output with respect to employment. In a Cobb-Douglas production function, elasticity is a constant. Weintraub mentions none of these things but displays a tremendous enthusiasm about the "magic" constant k. He proceeds to turn his definition $PQ/wN = k$ into the behavior hypothesis

$$P = kw\frac{N}{Q} \tag{2.1}$$

To do this, he must clearly assume that P is a dependent variable and that w and N/Q are both independent variables. As we shall see later, these assumptions are crucial for his policy conclusions.

On page 69 Weintraub is ready for what he calls "the wedding ceremony," i. e., the inclusion of the law of the "magic" k into a macroeconomic model. The ceremony is performed within a 6-page chapter which is called "the only chapter that is directed primarily to specialists in economics." The specialist, I am afraid, will benefit no more from this chapter than he does from the others. A Keynesian-type macroeconomic model of nine equations is set up and is said to determine "nine" unknowns. Unfortunately, the list of unknowns contains eleven items. Things are not quite as bad as the casual reader may fear. It turns out that k has been included among the unknowns; and k is surely the queen of the book's parameters. So there is one unknown too many. Furthermore, there is a verbal afterthought to the effect that the supply of money M must be equal to the demand for money L. This provides the unwritten tenth equation. The list is still quite confusing, however, for N (employment) and Q (physical output) are not included in terms of their absolute values but merely in the form of the ratio Q/N. Weintraub cannot mean this literally: he wants to determine absolute employment and output. The confusion is heightened further by the use of R for the unit wage cost on pages 9 and 48 and the use of R for total profits on page 72. But suppose the reader fights his way through the confusion, what does he find? He finds no discussion of the existence of a solution, let alone a discussion of its properties it if exists. Nor is the model used for anything in the later chapters. Even if its formal deficiences are repaired the model seems to add nothing to the model presented so lucidly, verbally as well as mathematically, by Lawrence R. Klein in his *Keynesian Revolution* thirteen years ago. One of the many merits of the Klein model is the care with which it supplies the additional information needed to determine the absolute level of prices or wages in addition to what is needed to determine the system in terms of real quantities.

What are the policy conclusions of Weintraub's book? There are two of them. First, inflation being wage-push inflation, the Federal Reserve system cannot fight it without creating unemployment. Second, inflation being wage-push inflation it might be restrained if a Federal agency would collect data on it and show them to Congress. The mere threat of informed congressional action might well, Weintraub thinks, "counsel restraint." Whatever one may think of the policy conclusions from the point

of view of policy, they do not follow from the analysis of the book. Constancy of k may mean that once w is set P follows, *or* it may mean that once P is set w follows. Weintraub clearly favors the first interpretation, but he offers no reasons why he thinks it is superior; worse still, he does not ever seem to have been aware of the existence of the second possibility.

After chapters on income distribution and growth, reworking the "magic" k a few more times, the book ends in a peculiar Concluding Note. Here, new studies are promised "to demonstrate that the price level has behaved over the longer past in the way our basic growth formula suggests." And the closing sentence is this: "Considering that the formulation descends from the basic truism, I think we are entitled to be confident of the results." But if the formulation descends exclusively from a truism, then how could it *not* be true, and if so, why waste time testing it? On the other hand, if it also descends from some forebears *other than* the truism, how can we be confident? This methodological confusion is, I am afraid, typical of the entire book.

In defense of the book one might say that since it was intended for the layman, no really powerful techniques of analysis, statistical or theoretical, could be expected to be used in it. Well and good, but in that case the author should not have permitted his publisher to make such wild claims as appear on the dust cover. Nor should he have permitted himself to make similar claims in the text, comparing his law of the "magic" k and its importance to that of Newton's gravitational constant.

How American Buying Habits Change. *United States Department of Labor.* Washington, D. C.: 1959. Pp. xiv, 253. $1.00.

EVA MUELLER, *University of Michigan*

THE purpose of this monograph is two-fold. One aim is to bring together the results of six major surveys of consumer expenditures undertaken between 1874–5 and 1950. The first of these surveys was conducted by the Massachusetts Department of Labor, the other five by the U. S. Bureau of Labor, later the U. S. Department of Labor. Twenty-five detailed tables, some additional text tables, and nine charts are presented in an attempt "to make readily visible the main outlines of the several surveys." This is not a source book, but a convenient summary of major data.

The second aim of the monograph is to describe the improvements in living standards which American workers have achieved since 1888. Unfortunately, this second task is approached only in part by making quantitative comparisons based on the expenditure surveys. In an apparent attempt to make the monograph interesting and lively for the general reader a great deal of miscellaneous descriptive information is added. There are quotations from case studies dealing with the living and working conditions of immigrants around 1900, accounts of changes in legislation pertaining to housing, sanitation, working hours, social insurance, etc. There also are discussions of major developments in public health, the growing importance of synthetic textiles and of detergents, labor saving devices around the house, the movement of working class families to the suburbs, and the like. One wonders whether the reader who is interested in the quantitative comparisons will see in the descriptive material anything but a tedious recital of well known facts.

The long descriptive text also suffers from annoying exaggeration. When speaking of workers at the turn of the century, much emphasis is put on the conditions under which slum dwellers and immigrants were forced to live. When the authors turn to the 1950's we hear of wage earners who no longer live in an identifiable working class neighborhood, are officers of the PTA, go to the same dentist as the salaried groups, own air-conditioners, power lawnmowers, outboard motors, and home workshops. Practically nothing is reported about the more prosperous segment of the working

population around 1900, or about slum dwellers in 1950. The Puerto Rican immigrant of today and the worker who lives in his own air-conditioned house are worlds apart in their consumption levels. The problem of diversity within the wage earner group is buried under excessive emphasis on averages.

The chapter headings suggest the scope as well as the tone of the volume. Chapter I, "The Bell and the Bay Window," paints a picture of the great material advance of American workers without reference to the statistical data. Chapter II provides an over-view of changes in family income and expenditure patterns as revealed by the six expenditure surveys. The next seven chapters deal with particular areas of consumption: housing ("From the Slums to Suburbia"), food ("Meals, Menus and Market Baskets"), household operation, clothing and personal care, health care, transportation, and leisure. The last chapter compares the scope and method of all expenditure surveys of the Department of Labor. For the non-expert interested in the expenditure data or the development of survey techniques, this is a valuable summary; for the professional expert a recent monograph by Helen H. Lamale goes over the same ground in more detail.

In summarizing the expenditure surveys, the monograph performs a distinct service. The material presented could be brought together by individual students of consumer economics only at the cost of considerable time spent in digging back into the original data. By contrast, the researcher interested, say, in exploring changes in housing conditions would hardly find the data presented sufficient; the monograph is not intended for him. The bulk of the statistical material consists of tables showing changes in the distribution of expenditures between categories of consumer goods (major and minor) over time and differences between income groups at a given point of time. Occasionally there are tables presenting the trend of deflated dollar amounts of expenditures over the years. No adjustments are attempted for changes in the population groups covered by the surveys, but the text contains some references to this problem. Break-downs by such variables as family size, region of the country, or occupation are rare, but appear where they are most relevant. There are, in addition, data on changes in quantities of various foods consumed; changes in number of various garments purchased by men, women, and children in different parts of the country; changes in the distribution of medical care expenditures by type of expenditure; uses made of the car by different occupation groups. Finally, there is an interesting table which shows the change in standard family budgets over time (pp. 236–8).

The scope of the monograph is to a large extent determined by the six expenditure surveys. Only sporadic use is made of data from other sources. For example, practically nothing is said about amounts saved or forms of saving—probably because of the weakness of the savings data in the expenditure studies. No attempt is made to enrich the housing data by drawing on Census materials also. Advances in the economic welfare of U. S. workers might have been placed in some perspective by making comparisons with other nations or with the progress of other groups in the U. S.; but again, this would have required some presentation of "outside" data. The monograph depicts long term changes from one expenditure survey to the next; cyclical fluctuations in consumption patterns again fall beyond the scope of the data. In brief, this is a study to summarize a body of data rather than a study which searches for the factors which promote and give direction to changes in consumption.

There is no point in criticizing the monograph for its limited purpose. The idea of writing economic history not in terms of changes in economic institutions and legislation, but in terms of changes in the fortunes and behavior of consumers is challenging. The BLS monograph points up the need for economic histories that focus on people and on major analytical variables such as consumption, investment, or the impact of

the government sector. It is to be hoped that this monograph will stimulate other histories of consumption, less restricted in their use of data and more analytical in intent. To write such a book obviously is not the task of the U. S. Department of Labor.

The Dynamics of Supply: Estimation of Farmers' Response to Price. *Marc Nerlove.* Baltimore: Johns Hopkins Press, 1958. Pp. ix, 268. $5.00, Cloth; $4.00, Paper.

ROBERT L. GUSTAFSON, *Michigan State University*

THE economist who wishes to apply theory to fact is faced with the necessity of estimating the parameters of functional relationships, and is plagued by the inadequacy of the available means by which to make such estimates. A large amount of research effort has been devoted to improving the means, and this study by Marc Nerlove constitutes a significant contribution to that effort. The general characteristics of this contribution are similar to others: (1) The basic notion is the fitting of hyper-planes to observed data or functions of observed data (in essence, some form of "least squares"); (2) Demonstration is made that if the fitting is done in some currently customary way, the conclusions are likely to be erroneous; (3) More sophisticated methods of fitting are introduced, corresponding to what are thought to be crucial aspects of the problem at hand (in the present case, "price-expectation formation" and "lags in adjustment") and making use of concepts from accepted statistical methodology (e.g., maximum likelihood); (4) Finally, and especially if some empirical applications are made (as they are in this case), it is pointed out that, while the new methods may offer some improvement over what was before available, they—perhaps inevitably—raise more questions than they answer.

The book consists of (1) two chapters on "Theoretical Considerations," the relevance of some of which does not appear until considerably later; (2) a chapter surveying some previous studies on agricultural supply functions; (3) four chapters essentially on the nature of the data; (4) two concluding chapters on "Statistical Analyses." The three commodities whose supply functions are studied are corn, cotton, and wheat, in the U. S., and the "dependent" variable in all cases is acreage, rather than the actual quantity of the commodity supplied.

The book may be recommended, not only to those who have a substantive interest in its specific subject-matter, but also to research workers in general in economics and statistics, as a good example of (a) the adaptation of statistical methodology to a problem area of considerable general significance, and (b) the careful application of the methodology to specific data. Chapters V–VII are of interest primarily as a description of some aspects of U. S. agriculture.

In recent years a number of applications of essentially the kind of method which Nerlove refers to as the "non-iterative" have been made, both within and outside the field of agricultural economics, and in estimating relationships other than supply; in some cases with seemingly quite "good" results. The general approach would seem to be, at least conceptually or potentially, applicable in a variety of situations where "expectations" and/or "lags" are relevant. The nature of Nerlove's model, in its simplest form, is as follows: The supply function is

$$x_t = a_0 + a_1 P_t^* + u_t \tag{1}$$

where x_t is acreage, a_0 and a_1 are parameters, P_t^* is expected "normal" price (not observable), and u_t is the disturbance. The price-expectation relation is:

$$P_t^* = P_{t-1}^* + b(P_{t-1} - P_{t-1}^*) \tag{2}$$

where P_{t-1} is actual price and b is the "coefficient of expectation," $0 < b \leq 1$. To estimate a_0, a_1 and b, two alternative procedures are used:

(I) The "iterative" method: Equation (2) is shown to imply that P_t^* is (approximately) a weighted average of actual past prices,

$$P_t^* = \sum_{i=0}^{t} b(1-b)^{t-i}P_{i-1} \tag{3}$$

where S denotes sum. This expression is substituted into (1), and least squares estimates (which are maximum likelihood under the usual kinds of assumptions) are obtained by a numerical, iterative procedure. As is often the case in econometrics, little is known about the small-sample properties of the estimates.

(II) The "non-iterative" method: Equation (2) is substituted into (1), and the result reduced to:

$$x_t = b_0 + b_1P_{t-1} + b_2x_{t-1} + v_t \tag{4}$$

where

$$\begin{cases} b_0 = ba_0 \\ b_1 = ba_1 \\ b_2 = 1-b \\ v_t = u_t - (1-b)u_{t-1} \end{cases} \tag{5}$$

The b_i's are estimated by least squares regression applied to (4), and the estimates of the parameters of interest then obtained from (5).

The estimates of the elasticity of acreage with respect to expected price (at the mean values of the variables), by the iterative and non-iterative methods respectively, were: for cotton, 4.53 and 0.67; for wheat, 1.18 and 0.93; and for corn, 0.35 and 0.18. (These were based on the simplest model, similar to the one outlined above except that a trend term was included. The iterative method was not used on the equations which included more variables.) One of the main sources of difficulty is that if the u_t are independent of each other, then the v_t cannot be, and vice versa. Use of weighted averages exacerbates the problems caused by serial correlation in the disturbances, since the averages (P_t^*) will tend to be serially correlated even if the original prices were not. (If the P_t were randomly and independently distributed with constant variance, then the serial correlation coefficient of the P_t^* in an infinite sample would be $1-b$.) Intuitively at least, any possible resulting bias would appear to be maximized when the weights (i.e., the value of b) are chosen not independently of the sample, but so as to maximize the correlation in the sample between x and P^*.

An alternative specification of the simplest model is

$$x_t^* = a_0 + a_1P_t^* + u_t \tag{6}$$

where x_t^* is "desired" or "equilibrium" acreage (not observable);

$$P_t^* = P_{t-1} \qquad \text{(i.e., } b = 1\text{)} \tag{7}$$

and

$$x_t = x_{t-1} + c(x_t^* - x_{t-1}) + w_t \tag{8}$$

where c is the "coefficient of adjustment" and w_t is a disturbance. Substituting (6) and (7) into (8) gives again (4), with the substitution of c for b in (5) and with $v_t = cu_t + w_t$. I.e., the two alternative specifications lead to the same estimates (by the "non-iterative" method), but with different interpretations.

The writer has not noticed any major "errors" in Nerlove's study, but, of course, simplifying assumptions are necessarily made. Some omissions which might be worth keeping in mind when interpreting the results, or when contemplating further research in the area are:

1) No attempt is made to consider the "simultaneous equations" problem, which may be important in the case of a commodity for which the quantity demanded, as well as the quantity supplied, depends on past prices. Of the three commodities considered, corn, at least, would appear to have a demand function of this nature, since the level of production of hogs, and hence the demand for corn, is largely influenced by past hog and corn prices. (And it may be noted that Nerlove has the most trouble with his regressions for corn.) It is true that at least part of this particular difficulty may be obviated by the use of acreage rather than quantity supplied as the dependent variable. Even if this is the case, however, the discussion of stability of market equilibrium would likely be modified by consideration of lags on the demand side.

2) Nerlove points out that if an adjustment lag is included in the model *as well as* a price-expectation coefficient not equal to one, then acreage lagged two years as well as acreage lagged one year appears in the "reduced" equation with which the estimates are made (in the case of the "non-iterative" method). His regressions of Chapter VIII, however, include only acreage lagged one year, and he points out (in Chapter IX) that the resulting estimates of the long-run elasticity of supply with respect to expected price will generally be biased (if, in fact, *both* influences are operative). His conclusion about the probable *direction* of the bias appears in error, however. Using his notation (footnote 1, page 241), he has $E\hat{\pi}_2 < \pi_2$, which is all right, but the relevant question is whether $E\hat{\pi}_2 < \pi_2 + \pi_3$, and this can be true only if $q_2 > 1$, which appears highly unlikely. (This is a relevant quibble, since one of the main conclusions of the study is that elasticities of supply are probably higher than they have sometimes been held to be.) Unfortunately, the attempt to introduce acreage lagged two years, and estimate both the expectation coefficient and the adjustment coefficient, in Chapter IX, met with little success. It might be mentioned that some other researchers, following Nerlove's lead and making similar attempts, on different data, have had similarly little success. The evidence suggests that the model, as developed so far, is not adequate to cope with this kind of fairly simple generalization.

3) "Expected normal price" is assumed to be completely determined by past actual prices and a constant coefficient of expectation, with no consideration of other possible influences. This seems to be a highly unrealistic restriction, and relaxation of it, at least to the extent of including a random disturbance in the price expectation relation (if better alternatives are not available) would seem to be worth considering in future applications or extensions of the model. It does not appear, however, that simply adding a disturbance term would, by itself, have any effect (under the usual kinds of assumptions) on the interpretation of Nerlove's own results.

The book contains a number of sometimes confusing typographical errors.

The Postwar Rise of Mortgage Companies. *Saul B. Klaman.* New York: National Bureau of Economic Research, 1959. Pp. xv, 102. $1.50.

JULIUS MARGOLIS, *University of California, Berkeley*

THE mortgage company is a firm whose principal activity is originating and servicing residential mortgage loans for institutional investors. It acts as an intermediary between the builder or home owner and the institutions with funds; it services the mortgage it originates; it relies on commercial bank credit to hold temporary inventories of mortgages. Its rapid rise during the postwar period reflects the shifts in the structure of the mortgage market which have given rise to the need of new specializations.

Mr. Klaman has examined in detail the brief history of the postwar rise of the mortgage companies. During this period the companies experienced a steady and rapid rate of growth. The major question asked by the study is: Why have they

grown? His answer can be put succinctly. The companies are the brokers between the rapidly growing institutional suppliers of credit, especially the insurance companies, and the home builders. The credit suppliers are national while the home builders are local and often located in regions distant from the fund suppliers. The introduction of the federally underwritten mortgage created the credit instrument which permitted the easy flow of institutional funds to small local investments. The mortgage company creates the contacts among these two groups. Though the development of this service is of interest another aspect of the study is potentially far more interesting.

Klaman presents a great deal of data about the mortgage company industry: the age distribution of companies, their sources of income, size classes, growth rates, asset structure, uses and sources of funds, geographic distribution and production characteristics. Some of the data were developed by adjustments of material in government files and others were developed from a special survey. The data do allow for testing of hypotheses about the growth of firms, economies of scale, adequacy of performance and other interesting questions which can be raised in an industry study. Klaman did not direct his attention to these questions. His analysis was focussed on an explanation of the development of the industry. His careful descriptive material can prove useful to others who might want to test hypotheses about the structure and performance of an industry.

An Introduction to Electronic Data Processing. *Roger Nett* and *Stanley A. Hetzler.* Glencoe, Illinois: The Free Press, 1959. Pp. 287. $6.75.

T. A. YANCEY, *University of Illinois*

THIS book is concerned with providing a comprehensive discussion of electronic data processing systems for the non-mathematically oriented beginner in data processing. The authors have drawn on their experience obtained in organizing and teaching in the data processing training program of the Air Force's Air Material Command to determine the appropriate material to include and the emphasis to be given each aspect of data processing. As a result, the book reads very much like a series of lectures might sound when delivered by a good expositor who enjoys teaching.

In the first chapter, the authors trace the history and development of automation, the rise of the scientific method, and the history and development of computers from the abacus to the modern general purpose automatic computer. This is done in an interesting if somewhat casual way. Having established the evolution of automation and computers in general, analog and digital computers are introduced, and their basic differences and advantages are described in non-technical terms by the use of analogies and examples. The book is concerned primarily with general purpose digital computers of which the most recent UNIVAC's and IBM 700 series are the primary examples. Models from these two series are referred to frequently in connection with programming and illustrations of computer components. The discussion of compoments of computers—control units, storage devices, input-output devices, and arithmetic units—includes all types of devices that have been used in the past as well as those most often incorporated in currently constructed computers. Most of this section is spent describing a variety of internal storage devices and input-output devices and indicating the advantages and disadvantages of each device.

Chapters III and IV are on binary arithmetic, machine instructions, programming, and computer logic. The topics, other than computer logic, are covered quickly in a total of forty pages of which only nineteen pages are devoted to flow charting, machine instructions, and programming. The book is definitely not at all suitable for any but the most sketchy notion of how to program a problem on a computer. As a text on

data processing, it would have to be supplemented heavily in this section if programming is to be taught at all. The section on computer logic impressed the reviewer as one that could well be omitted. Here, the authors elaborate on computer circuitry' some historical notes on discoveries of the behavior of electrons, and some very elementary descriptive physics. It is in this section that the jargon of the computer designer appears, for example, "gating," "flip-flops," etc. This is interesting general information but not necessary to understanding the use of computers.

The next two chapters are on computer applications listed sometimes by method of analysis, e.g., linear programming, and sometimes by area of application, e.g., applying computers to commercial problems. Most of the illustrations are brief and indicate little more than that it is possible to program a particular problem on a computer.

The remaining fifth of the material is devoted to problems involved in the installation of a large electronic data processing system in a business firm. The authors spell out a number of difficult problems in converting an organization to computer operation. There is a discussion of the necessary detailed plans and scheduling of steps in the conversion to computers, the training of personnel, the organization of the computer group, and the sociopsychological impact of introducing computers.

An appendix of over fifty pages of technical information is supplied to discuss the performance characteristics of a number of commercial computers. Most of this space is devoted to the products of the two giants in the field. Also, a useful bibliography is provided at the end of the book which will be particularly useful to any reader who desires more than the most cursory knowledge of computers and their application. The brief treatment of topics is not in itself a basis for criticism, but is due to the decision to present such a broad sweep of the field of data processing in a book of modest proportions. Considering the authors' aim to produce a very readable book on automatic computers for a very heterogeneous audience, they have not done badly.

Handbook of Automation Computation and Control. Volume I (1958) and Volume II (1959). *Eugene M. Grabbe, Simon Ramo,* and *Dean E. Wooldridge,* Eds. New York: John Wiley and Sons, Inc. $17.50 each volume.

MITCHELL O. LOCKS, *University of California, Los Angeles*

THIS set represents a milestone in the unification of knowledge. It is aimed primarily at automatic computation, data processing, and control engineering. However, because high speed computers are used in so many different branches of science and technology, the material covered includes a wide range of applied measurement and engineering activities. The result is an encyclopedia-type compendium on techniques of the use and design of data processing systems.

The *Handbook* is a planned 3-volume set of which the first two have appeared in print as of the date this review is being written. Volume I consists of 26 separate articles on what is essentially "Applied Mathematics "with most of the articles written by staff members of either the University of Michigan or the General Electric Company. Volume II consists of 31 chapters on Programming, Business Data Processing, and Computer Design and Engineering. Eleven of the authors of sections of Volume II are on the staff of either Thompson Ramo Wooldridge or its subsidiary, the Space Technology Laboratories, and the majority of the remainder of the authors come from firms or educational institutions in the Southern California or Boston areas. A notable achievement in this volume is the incorporation of a book-size 270-page chapter on "Programming and Coding" by John W. Carr III of the University of North Carolina.

Since Volume I deals largely with numerical operations while Volume II has to do with applications and technology, the former is of more interest to most statisticians

than the latter. Similarly, Volume III which will be on "Systems and Components" should not be of interest to many statisticians unless they have overlapping areas of responsibility. Therefore the remainder of this discussion will deal only with Volume I.

Of the articles in that volume, there are three that deal most directly with the subject matter of Statistics. These are "Probability" by A. H. Copeland and "Statistics" by A. B. Clarke, both of the University of Michigan, and "Digital techniques in statistical analysis of experiments" by Joseph M. Cameron of the National Bureau of Standards. All of these articles are competently written and contain much useful material. However, if information on those topics is desired, a statistician is likely to have available other sources of information. The real merit of the *Handbook* is that it provides a quick reference on other topics related to one's own work, but in which his library may not be extensive, and a bibliography telling him where to look further.

The other articles in Volume I which should be of the most interest to statisticians are the first 11 chapters on "General Mathematics," the separate chapters on "Numerical Analysis" and "Information Theory," and Dr. Arnoff's 124-page opus on "Operations Research."

In view of the difficulties in getting different authors to write about related matters of substance without at least in part duplicating each other, the editors are to be commended for assembling a wide variety of articles of quality by different writers without too much overlapping in subject matter. This *Handbook* should be useful to research workers and teachers in the computation and data processing arts for a number of years to come.

Fractional Factorial Experiment Design for Factors at Three Levels. *W. S. Connor* and *Marvin Zelen.* Washington, D. C.: National Bureau of Standards, 1959. Pp. v, 37. $.30. Paper.

FRANKLIN A. GRAYBILL, *Oklahoma State University*

THIS is a monograph published by the United States Department of Commerce in the National Bureau of Standards Applied Mathematics Series—54.

This monograph is primarily a catalogue of fractional factorial arrangements for 3^n factorial systems. The arrangements which are catalogued are 1/3, 1/9, 1/27, 1/81, and 1/343 replications for various combinations of factors which range from 4 to 10. There are also arrangements for grouping into various sizes of blocks. The fractional designs which are listed in this monograph are such that no main effects are aliased with other main effects or two factor interactions, and such that as few two factor interactions as possible are aliased with other two factor interactions. Each design is accompanied by a fundamental identity which can be used to show what effects are aliased. The first 10 pages is an exposition of the theory of the construction of the designs, explanation of the tables, and the analysis of variance. The remainder is the catalogue of the various fractional factorial designs.

This monograph seems to me to serve a useful purpose. For anyone interested in 3^n fractional factorial arrangements this monograph contains many of the patterns. However, even though the theory is explained, it is so condensed that it won't be too useful to anyone who is not already well acquainted with the theory of fractional factorial designs. I think that the authors never intended this to be an elementary treatment of the theory of fractional factorial designs, but included a resume of the theory for completeness and to better explain the actual catalogue of arrangements.

In general the explanations seem to be clear and well written. However, there is an unusual statement on page 7. It is "This model can be written as

$$y(x_1, x_2, x_3, \cdots) = \eta(x_1, x_2, x_3, \cdots) + \beta_j + \epsilon(x_1, x_2, x_3, \cdots)$$

where $y(x_1, x_2, x_3, \cdots)$ represents the actual observation of the factorial combination, $\eta(x_1, x_2, x_3, \cdots)$ represents the true value of the observation $\cdots$." I don't

understand why the authors chose the words "true value." It seems to me that it would convey more meaning if $\eta(x_1, x_2, x_3, \cdots)$ referred to the "treatment effect" or something similar.

Expectations, Uncertainty and Business Behaviour. *Mary Jean Bowman*, Ed. New York: A Conference Held at the Carnegie Institute of Technology, October 27–29, 1955, under the Auspices of the Committee on Business Enterprise Research of the Social Science Research Council, 1958. Pp. vii, 202 $2.00 (paper).

DOUGLAS G. HARTLE, *University of Toronto*

A CONFERENCE on research on expectations, uncertainty and business behavior was held at the Carnegie Institute in October, 1955, under the auspices of the Committee on Business Enterprise Research of the Social Science Research Council. This paperbacked volume contains the revised papers presented at the conference, some of the discussion which followed the papers, and a most useful Introduction written by Mary Jean Bowman who, in the words of the foreword, "developed the plans for the conference . . . obtained the participation of the authors . . . and obtained their collaboration in revising the papers . . . ."

As Bowman points out, it is impossible to briefly summarize the contents of all fourteen papers, two notes and the comments of a number of discussants which this volume contains, much less evaluate their significance. About all a reviewer can usefully do is to list the principal papers and give some indication of their contents.

The volume is made up of four parts. The first part consists of two papers and a note on the nature and structure of expectations. The first paper, by Nicholas Georgescu-Roegen, provides an analysis of the nature of the major theories of expectations, including the Classical, Frequentist, Carnapian and Subjectivist schools. It also includes an evaluation of their relationships to each other and to the development of a theory of rational decision making. The author concludes that it is impossible to derive an objective way of arriving at decisions which involve true uncertainty; and he apparently believes that ultimately such decisions must be based on "good judgment." The second paper, by G. L. S. Shackle, provides a brief restatement of his well-known theory of expectations and an attempt to develop an explanation of the advantages and disadvantages of liquidity in terms of this theory. The section concludes with a brief but interesting Note by Ward Edwards which contrasts Shackle's theory of surprise with a decision model based on non-additive subjective probabilities which Edwards and Donald Davidson are developing.

Part II consists of five papers on the "Psychosocial Approaches to the Study of Expectations and Business Behaviour." The first paper, by Herbert A. Simon, is entitled, "The Role of Expectation in an Adoptive or Behavior Model." This paper seems to me an important contribution to the field, for it mirrors all of the breadth of understanding, imagination and incisive analysis which we have come to expect from this author. Simon first distinguishes between what he terms programmed and non-programmed decisions. The former constitute a class of recurring decisions which are dealt with by a well-established, systematic procedure; the latter involve a choice selection in which there is "much stirring about, deliberation, discussion (and) often vacillation." Programmed decisions are what economists usually consider "rational decisions" and they are the types of decisions to which modern decision theories, of the Savage variety, apply. Non-programmed decisions, on the other hand, are what economists might term the process of innovation; for "the alternatives of choice are not given in advance"; the "means-ends connection between choice and consequences are imperfects known"; and a "frame of reference must be chosen before the choice can be brought within the limits of human computation."

Following a brief discussion of how social institutions stabilize expectations and widen the sphere of programmed decisions, Simon turns to a brief analysis of the

process of innovation or unprogrammed decision making. It is in this section of the paper that Simon's insights are most decisive, for by observing the remarkable similarity between the economists' concept of opportunity cost, and the psychologists' concept of aspiration level he is able to develop an explanation of the innovation process which may prove to be of the greatest significance in bringing these two disciplines together in an analysis of this important but difficult question.

The Simon paper is followed by a paper by George Katona, which discusses the role of expectation as part of ex ante data, the psychology of decision making, the origin of attitudes and the measurement of expectations. Two papers by psychologists G. Patrick Meredith and William E. Henry and a paper by a sociologist, Albert Lauterbach, complete this section. While the idiom of these contributions will probably be foreign to most of the readers of this journal, the framework provided by the Simon contribution makes them more readily understandable and pertinent.

The two final parts of the volume include papers with an applied orientation. Most of them attempt to devise or test models involving expectations, forecasts or plans. Part III is basically devoted to the analysis of expectations and time horizons in particular industries; while Part IV includes papers which are concerned with the forecasts or plans made by the establishments in the economy as they relate to changes in the aggregate level of economic activity.

Ruth Mack has presented the framework of a raw material inventory model involving expectations of future prices applicable to firms which are competitors in the factor market. While this model is developed with particular reference to the boot and shoe industry, with which the author is so familiar, it probably has wider applicability. Of the three attempts to relate the concept of expectations to empirical work at the industry level, this paper appears to be the most successful. Merton J. Peck in his paper on inventory behavior in the aluminum industry is at pains to show the relevence of marginal analysis to an understanding of business behavior under uncertainty; but it seems more successful as a description of inventory policy in this industry than as an illustration of the virtues of the marginal analysis. Hans Brem's paper deals with differences in the timing of price and non-price competition in the automobile industry. While an interesting analysis of oligopolistic behavior, it does not seem to be particularly germane to the subject matter of the conference.

On the whole the papers included in Part IV have perhaps less novelty than the other papers included in the volume. This judgment in part may be the result of the greater familiarity of the reviewer with the problems with which they are concerned, and in part because of prior publication of their results. The first paper, by Franco Modigliani and Kalman J. Cohen, is entitled: "The Significance and Uses of Ex-Ante Data."[1] This paper lays the foundation of the "realization function" approach to the use of ex ante data which Modigliani has since applied in the article written with Weingartner (which appeared in the February, 1958, issue of the *Quarterly Journal of Economics*). Because the authors are in the process of revising their basic work in this field readers of the article under review should bear in mind that it may no longer represent their current views, at least in detail.

The usefulness of their approach is attested by the success of Robert Eisner's paper on "Expectations, Plans and Capital Expenditures: A Synthesis of Ex Post and of Ex Post and Ex Ante Data," which is, in part, an application of the anticipations function–realization function concepts developed by Modigliani and Cohen. With the unique data he has been able to examine, which includes forecast and actual values of

[1] In the first footnote to this article the authors state that it provides a summary view of a "full and more rigorous exposition given in a paper entitled 'The Role of Anticipations and Plans in the Economy of the Firm and Their Uses in Economic Analysis and Forecasting'." The latter paper is now (February, 1960) being extensively revised and has not yet been published—the cited reference notwithstanding.

a number of variables, including capital expenditures, sales and profits, for the same firms over a number of years, Eisner has been able to test, in an admittedly tentative fashion, the Hicks-Harrod formulation of the acceleration principle. This has led Eisner to suggest some interesting modifications involving distributed lags. Economists interested in this field will applaud Eisner's promise to intensify his research in this direction in the future.

Irwin Friend's paper consists of a summary evaluation of the investment intentions surveys; and, to a lesser extent, forecast surveys of other variables. The passage of time has unfortunately dated this evaluation and it should now be revised to take into account several papers which were presented at the Universities–National Bureau Conference held at Princeton, November 8–9, 1957. These papers will soon be published.

The final paper in the volume by William Durcovich provides a neat "Evaluation of Some Naive Expectation Models for Agricultural Yields and Prices." Basically the author attempts to assess how well the farmer would be able to forecast these variables using extrapolative devices of increasing degrees of "sophistication," including techniques which would take into account information published in agricultural outlook reports. Although this paper undoubtedly provides some interesting information, it does not appear to be closely related either to the other papers or to the principal purpose of the volume.

The wide disparities in approach among the papers included within the various Parts; and the immense differences in subject matter between the Parts themselves, provide a measure of our current understanding of the relationships among expectations, uncertainty and business behavior. Much of this complex area is still largely undeveloped, yet this conference undoubtedly elicited some important contributions. Moreover, drawing them together serves a useful function; for it becomes more feasible to assess the gaps and inconsistencies in our knowledge and helps to indicate research avenues which might usefully be pursued. While this purpose of the volume ultimately can be served only by a careful study of the papers themselves, the perceptive Introduction provided by Mary Jean Bowman will prove of great assistance to those who wish to evaluate the weaknesses and highlight the strength of our present understanding.

The Federal Funds Market: A Study by a Federal Reserve System Committee. *Board of Governors of Federal Reserve System*, 1959.

EUGENE C. ZORN, JR., *Dallas, Texas*

THIS work of a Federal Reserve System Committee contains data and other information that are obviously valuable for the formulation and execution of monetary policy. The committee, in addition, has performed a genuine service for the banking community, money market observers and operators, and students of money and banking. Facts concerning the Federal funds market have been scattered and incomplete. Their assembly into a well-organized and highly informative booklet was long overdue.

The progressive development of flexible monetary policy over the past decade has complicated the process of money market analysis. New instruments and techniques have been developed and others have been taken down from the shelves and dusted off. Concepts and traditions that once seemed permanent have been modified or disappeared. Flexible monetary policy has not been thrust upon us suddenly. We have had to grow up to it. Part of this growing-up process has been the gaining of knowledge and experience regarding the market process and the forces of supply and demand that shape such a process.

That is the story of the Federal funds market. In recent years it has become one of the most interesting and significant elements in the whole complex of money market analysis and policy administration. Not too long ago detailed knowledge of Federal funds operations was confined to a relatively few specialists at the trading desks of dealers, the money desks of banks, and within the Federal Reserve System. But the increasing sensitivity of the money market under conditions of restraint is making it more essential for managers of short-term investment funds to understand the intricacies of all facets of their field. The System Committee has given us a good set of working tools to add to the fund of knowledge for this purpose.

Apart from the descriptive information on Federal funds operations, the Fed has plowed fertile ground in providing data as of November 1956 on the size of the Federal funds market, the types of transactions, their geographic distribution, and rates.

Three years have elapsed since the date of its survey. Deposits are turning over more rapidly, the discount and market rates are significantly higher, and Treasury bills—a money market instrument that competes with Federal funds—have increased substantially in amount, and have come to rest in the hands of large non-bank investors who are gaining a sophistication regarding money market analysis. Developments such as these give rise to new relationships within the money market. To help interpret their significance, a continuous flow of information is highly desirable. The broader the understanding, the more perfect and smooth the market process can be in reflecting basic forces of supply and demand. It is hoped, therefore, that the Federal Reserve will not allow too long a period to elapse before making another survey giving us a profile of the Federal funds market.

Comparisons of the United States and Soviet Economies. *Joint Economic Committee.* Congress of the United States. 3 parts (616 pages; $1.70); and a volume of Hearings (292 pages; 70 cents). September 1959–January 1960. Government Printing Office, Washington, D. C.

LEON M. HERMAN, *Library of Congress*

THIS series of publications grew out of a skillful, well-timed enlistment by the Joint Economic Committee of the expertise of a group of working economists for the task of summing up their latest findings on the performance of the Soviet economy and discussing their implications for national policy-making in the United States.

The first three volumes consist of papers written by the individual economists in advance of the hearings, by way of communicating with the other participants in the symposium. These papers are grouped into sets of panels by broad subject-matter: industry, agriculture, national income, etc. The fourth volume carries the summary statements made by the panelists during their appearance as witnesses before the Joint Economic Committee and the discussions that followed.

While the orderly division of the papers by panel, or by volume, is at times upset by reason of late arrival of papers, the contributors to these volumes fall into three distinct groups. The largest group is made up of analysts working in the field of Soviet economics. Between them, they present a documented, comprehensive review of recent developments in the Soviet economy, offering wherever possible a framework within which Soviet performance can be realistically compared to that of the United States. This group accounts for eight out of the total of ten panels taking part in the symposium.

The ninth panel consists of representatives of non-governmental policy-making organizations (American Farm Bureau, National Education Association, Committee for Economic Development, etc.), who discuss the challenge represented by the rising economic capabilities of the Soviet Union in the spheres of the economy with which they are familiar.

The final panel, by far the most interesting for the general reader, consists of a group of "elder statesmen," seasoned observers of the Soviet world-wide drive for power, who summarize the salient facts placed on the record by the sector specialists and offer their evaluation of where we stand in the economic race with Moscow. Most of the attention of this group is, however, devoted to the important unfinished business that lies ahead if we are to avoid seeing our power eroded and the economic viability of the free world sacrificed by the failure to use in full the resources at our disposal.

Above all, this remarkable symposium has the great advantage of timeliness. The vast and varied body of facts it has assembled, having been written down by the working analysts between May and September in 1959, are the fresh data on the basis of which public thinking on the subject will be governed for the months ahead. The business of commercial publishing being what it is, it is a fair guess that some of these facts will probably turn up in the textbooks some two or three years hence.

The ideas on what are the alternatives for United States policy, domestic and foreign, stimulated by the findings of these investigations among the four members of the final panel, are similarly fresh and operative. A good representation of these ideas is mirrored in the highly perceptive paper of W. W. Rostow (part III), who discusses with the extra sense of perception of the economic historian the serious adjustments we must make, in our policy-making in general and in our allocation policy in particular, for coping with the type of long-term, unspectacular threat of erosion to our power and to the survival of democratic political institutions represented by the planned, inexorable expansion of the power of the Soviet Union.

An Econometric Model of Postwar State Industrial Development. *W. R. Thompson* and *J. M. Mattila.* Detroit: Wayne State University Press, 1959. Pp. vii, 116. $3.00.

WILLIAM A. CROMARTY, *Connell and Company*

THE authors of this book have tackled two difficult problems, (1) the theoretical problem of constructing an economic model which adequately explains industrial development, and (2) the equally difficult task of selecting estimators which give quantitative estimates of the economic relationships. Unfortunately, neither problem is successfully completed.

The authors decide upon the "estimated average annual change in the number of workers employed, by states" as the variable to be explained, measured for the years 1947–1954 inclusive.

The independent or explanatory variables are taken to be specific indicators which fall in the broad categories of,

(a) growth of the local market,
(b) factors reflecting the degree of industrialization,
(c) factors reflecting the level of state and local taxes,
(d) factors reflecting characteristics of the labor market,
(e) factors expressing the educational level of the population, and
(f) patents granted, expenditures, and employment in 1947 for specific industries.

The selection of the dependent variable as an indicator of industrial development seems to rest on the fact that data were available. Such a criterion is weak. Employment, as they point out, is only one variable which measures industrial development and it is questionable if it is a good overall indicator. For instance, for the period 1947–54, durable goods sales for the United States rose from 6.7 to 11.2 billion dollars, almost a 70% increase; or in physical terms the index of industrial production for durable goods rose 38 points from 1947 to 1954. However, the number of workers in durable goods manufacturing showed a four per cent change, going from 15,321,000 to 15,995,000. A more significant example, although not included in the text, is the case

of agriculture. Farm employment dropped from 10.4 million in 1947 to 8.6 in 1954. At the same time farm output on a physical production index basis jumped from 95 in 1947 to 109 in 1954.

Of an equally serious nature is the violation of the assumption that in least squares regression analyses, which the authors choose to use, the dependent variable is made up of observable data. Yet the footnote on page 89 indicates that some of the data were not available and had to be estimated.

Many questions could also be raised with regard to the selection of the independent variables. Personal income, for what is almost a current period, i.e. 1945–8, does not appear to be a good determinant of the number of workers employed. The reverse relationship would appear more plausible, which suggests that the treatment of this problem on the basis of single equation models may be too simplified for accurate results.

The authors appear to have trouble explaining the positive relationship between average hourly earnings and the number of workers employed. In reality it may be a very realistic relationship, except that the equation represents a supply relationship. This might be considered as an omission of identification procedures.

The estimating techniques are simple—correlation coefficients and linear regression analysis. The impression is left on page 35 that the technique itself selects the particular variables to be inserted into each equation and in some way provide sufficient estimates. The criterion of selection is not stated, however, and if it was on the basis of the values of the simple correlation coefficients the criterion is again very weak.

In all fairness to the authors it should be pointed out that many of the inadequacies of the study are apparent to them and they have pointed out some of these in Chapter V. Their interpretation of the regression analysis results between industry classifications is commendable and probably the most useful section of the book, and a good summary of their findings is presented in Chapter VI. The book will probably be useful to persons involved in state planning in spite of the statistical and economic shortcomings pointed out previously.

Operations Research: Methods and Problems. *Maurice Sasieni, Arthur Yaspan,* and *Lawrence Friedman.* New York: John Wiley & Sons, Inc., 1959. Pp. xi, 316. $8.50.

HARVEY M. WAGNER, *Stanford University* and *The RAND Corporation*

THE instructor of an elementary course in operations research (OR) expects ultimately that his students achieve the level of vaguely understanding the *A, B, C*'s of probability, sampling, inventory analysis (*IA*), replacement policy (*RP*), waiting line theory (*WL*, or more briefly *Q*), game theory (*GT*), linear programming (*LP*), sequencing of operations, and dynamic programming (*DP*). The textbook by M. Sasieni, A. Yaspan, and L. Friedman (whose initials we rearrange as a mnemonic abbreviation to *MYLAFS*) can be recommended for this purpose, as these men provide a chapter on each of the subjects, and for the most part spell out the material at the aforementioned level.

The readers are assumed to know the calculus, hopefully to have mastered "a modicum of statistics . . . equivalent to . . . a one semester course,"* to be graduates or advanced undergraduates, and to wish to "develop skill in formulating and solving mathematical models" within one semester.

In each section the authors, who developed the course material at Case Institute of Technology, admirably furnish a discussion of general theory and certain computational techniques, several completely solved problems demonstrating the techniques, and student's exercises; further each chapter is "complete in itself as a teaching unit."

* The students' memories are refreshed by the chapters on probability and sampling.

Evaluating the book in absolute terms, this reviewer, who has used the text in a class exactly meeting the prerequisites stated, feels that by and large the material is too easy and the scope too limited. Needless to say, the students were not in perfect agreement with this conclusion. In relative terms, the book compares most favorably with other competing volumes.

A sample of a few specific comments is in order. The *LP* chapter is about the best insofar as breadth of coverage is concerned. Little tie-in is made between the chapters on *IA*, *RP*, and *DP*. In the *Q* chapter, Erlang distributions are cited; in the probability chapter, the gamma distribution is described; but no connection between the two is mentioned. The authors deserve a mixture of kudos and brickbats for their *GT* section: the examples and problems are challenging. But my students failed to fathom from the text material what exactly a maximin strategy was in simple two-person zero-sum games (e.g., the class even had difficulty in verifying such strategies when they were stated numerically for particular games), and were misled as to how to compute algebraically such strategies. The *GT* chapter precedes the *LP* chapter, so the authors do not mention *LP* when discussing *GT;* this situation is not remedied in the *LP* chapter. Although the authors in Chapter 3 argue for the usefulness of statistical methods in *OR*, they noticeably sidestep demonstrating the relevance of such techniques at the crucial points in subsequent chapters. In short, many of the deficiencies stem from the authors' well kept promise of making each chapter complete in itself as a teaching unit.

Of course the imperfections in textbooks usually serve their purpose in providing material for lectures, and in recommending the book, this reviewer notes the ease he discovered in augmenting *MYLAFS* in the classroom.

Yearbook of Fishery Statistics, Vol. IX, 1958. *United Nations, Food and Agriculture Organization.* New York: Columbia University Press, 1960. Pp. xxix, 89. $4.50.

THIS volume continues the publication of data on catch and disposition of fish and fishery products throughout the world. Estimates for countries and species are presented. Results of a thorough revision of statistics on products by Japanese authorities are incorporated. Designations of material in tables are in English and French, textual material is also in Spanish. C. H.

Consumer Expectations, Plans, and Purchases: A Progress Report. *F. Thomas Juster.* National Bureau of Economic Research Occasional Paper 70. New York, National Bureau of Economic Research, Inc., 1959. Pp. xi, 171. $2.50. Paper.

MORRIS COHEN, *National Industrial Conference Board*

THE past fifteen years have been marked by a considerable interest in expectational research. The underlying thesis is a simple one. Each of the major economic sectors plans future activity. If these plans can be tapped and properly interpreted, then greater insights will be gained into subsequent behavior. Businessmen plan ahead for capital goods spending through capital budgets and capital appropriations, and governments plan ahead through annual budgets. Both business and government plans are generally formalized in written documents. Consumers also plan, but mostly in an informal manner, and their plans must be obtained either through personal interviews or mail surveys.

The pioneering work of George Katona and the University of Michigan Survey Research Center in developing consumer surveys focussed on consumer attitudes and buying plans is well known. For years, they were the only group continuing to make and publish studies on consumer expectations' research. Nevertheless, there were still some doubts. The 1955 Report of the Consultant Committee on *Consumer Survey*

Statistics organized by the Federal Reserve Board (one of a series of five distinguished reports too long neglected by the profession) paid homage to these pioneers, but still recommended that additional investigators in other institutions study consumer behavior through expectational data.

Mr. Juster's report represents a major methodological advance precisely of the type which the Federal Reserve Committee wanted. It is based upon mail survey data which are completely independent of the Michigan Survey Research Center. Subscribers to *Consumer Reports* published by Consumers Union of the United States, a nonprofit testing and rating organization, reply evidently in very large numbers to questionnaires. It is admitted, of course, that such a survey is not representative of all the nation's households; it is not even a random sample of the magazine's own subscribers. The results that emanate from this nonrandom survey may still prove to be of great methodological significance. They may not prove to be useful for business forecasting, though Juster presents evidence that they have been in the past, but they may well provide some underpinnings for a better theory and understanding of consumer behavior.

There is some merit to Juster's implicit foundation of his research: if you want to study consumer buying plans in depth, then ask the people who are most likely to plan. Subscribers to the activities of Consumers Union probably represent the most planning-conscious group of consumers in the country. What is missing most in this approach, of course, are the consumers who buy on impulse. The impulse buyers, those who purchase without a plan, can better be investigated through a survey of the entire population. Some question also remains about the representativeness of the CU planners. They may prove to be a different type of planner not representative of all consumers who plan. That is why in the final analysis the Juster results must be confirmed through random probability surveys.

The broad results of this progress report represent additional evidence on the usefulness of consumer buying plans and attitudes as a predictive tool. For example, it was found that from 1949 through 1957, the year-to-year changes in Consumers Union subscribers' buying plans for automobiles and major appliances foreshadowed actual purchases for the country as a whole. Taking income into account, fully 95% of the annual variations in these purchases were statistically explained by two variables—changes in disposable income and buying plans. Clearly, further efforts and additional research in buying plans are warranted.

However, the bulk of Juster's present research report centered on 25,000 re-interviews in April, 1958, of respondents to an earlier survey in October, 1957. This represented the largest re-interview samples ever taken which were concerned with consumer buying plans. Only with samples of this size can all the interrelationships be studied, holding various factors constant while other variables are analyzed.

One of the more important findings of the study relates to the time horizon of the planning period. By breaking down his basic 25,000 re-interviews for April 1958 into five subsamples, Juster is able to introduce experimentation into the field of consumer expectations. He found, as has the writer working with other materials, that "the level of buying plans reported in a given survey is sensitive to variations in the time horizon covered by the question asked." Further, buying plans for the next six months are "more apt to be meaningful indicators of future actions by households and are probably more useful as ingredients in a forecasting model." Hitherto, most consumer buying plan surveys have used a twelve-months' planning horizon. The Juster evidence thus provides further support for the more recent trend toward framing consumer buying plan questions with a shorter time span. Both the Conference Board's

continuing survey of consumer buying plans and the Census Bureau's still unpublished surveys in this field have been using the shorter planning period.

As far as expectational variables are concerned, perhaps the most provocative finding was the significance of household expectations about personal financial prospects over a five-year future period. These were among the most powerful influences affecting both plans and purchases. Since presumably such expectations are less likely to fluctuate over the short-run, it would appear that some magic formula has emerged which can identify the more accurate planners. It is recognized that such identification may not be useful for aggregate predictions, since these expectations might be stable over time. Nonetheless, if this result could be substantiated it would represent a major contribution toward a better understanding of consumer behavior. This example illustrates the uncertainty about many of the important findings from this study and reflects the lingering doubts about the general applicability of results from nonrandom surveys.

Since most of the report is concerned with relationships between surveys of the same consumer units taken in October, 1957 and April, 1958, the whole question of seasonality in buying plans and attitudes must be raised. In The Conference Board's survey of consumer buying plans, a random, telephone survey of 5,000 households taken each month, it has been found that buying plans and some attitudes are marked by sizeable seasonal fluctuations. Furthermore, the timing of the Consumers Union surveys was intertwined with the 1957–8 business recession. Both cyclical and seasonal factors affect consumer buying plans and behavior, perhaps even the carrying out of the planned purchases. Any results that emerge from two observations in time, particularly those at cyclical turning points, must await further confirmation from additional research.

Professor Albert Hart, under the sponsorship of the Ford Foundation, is continuing the effort forward in time, using the CU survey panel, and his further results are eagerly awaited. Still, all investigators in the field of consumer behavior and particularly those specializing in consumer buying plans and attitudes will, I am sure, want to consult the many fruitful hypotheses and conclusions presented in this careful study.

The Population of the United States. *Donald J. Bogue.* Glencoe, Illinois: The Free Press, 1959. Pp. xix, 873. $17.50.

WALTER B. WATSON, *University of Washington*

THIS ambitious volume is the fourteenth in a series on population distribution by Bogue and others of Scripps Foundation and the Population Research and Training Center of the University of Chicago. The author maintains the high standards set by the earlier monographs in a work of truly staggering proportions. Twenty-six chapters and 701 double column over-sized pages include 384 tables and 93 figures, graphs, and charts. In addition the text contains numerous small unnumbered summary tables. There are also a 68-page statistical appendix, a 69-page tabular supplement to the chapter on occupational composition giving numerous demographic correlates of more than 400 specific occupations, and a similar 21-page supplement to the industrial composition chapter for 77 industries.

The author opens with an account of the size and growth of the population and closes with a discussion of characteristics and implications of the future population. In between are four excellent chapters on distribution (urban-rural, metropolitan-nonmetropolitan, states and regions, and the system of economic areas), one each on mortality and fertility (the latter by Wilson Grabill), two on migration (international

and internal), five on economic characteristics (labor force, occupation, industry, unemployment, and income), ten on other characteristics (age, sex, race-nativity, marriage, households and families, housing, education, institutional population, health, and religion), and one chapter on Alaska and Hawaii. The chapters are minimally interrelated and hang together only in terms of the content of the census and other available data rather than in terms of an underlying theory or point of view; this comment, however, is more a commentary on the character and state of advancement of the field than a criticism of the author.

The use of age as a control throughout and sections on metropolitan areas in several of the characteristics chapters are particularly well done. Grabill's chapter on fertility is good but contains an unfortunate and unnecessary use of the value laden term "welcome" in a reference to the postwar baby boom. (p. 302) International comparisons in several chapters provide a helpful comparative setting within which to view the United States population.

"The book is intended primarily for reference use, and has been written and indexed with this in mind." (p. *iv*) The excellent tables and illustrations and the lucid text are a major contribution in this regard. Many of the data are taken from standard census sources including various Special Reports and from the vital statistics publications, but the book is liberally and effectively supplemented and brought up-to-date by the inclusion of data from the Current Population Survey. The latter help to achieve the formidable task of "analyzing and interpreting the results of the 1960 census before it was taken." (p. *iii*) The reference value of the volume is further enhanced by the inclusion of data from the National Health Survey in a chapter on illness, data from the National Opinion Research Center in the chapter on religion (particularly in view of the absence of information on religion in the 1960 census), data from the National Housing Inventory in a housing chapter, and some unpublished data and special tabulations by the author. Introductions to the various chapters are adequate; the chapter summaries might well be expanded somewhat.

Frequent small typographic and editorial errors in text and tables detract from the otherwise excellent reference value of the book as does the absence of a bibliography. Small errors, such as failure to indicate that the numbers are in thousands in some of the tables, and no bibliography are unlikely to trouble professional demographers, but they may bother students, laymen, and even scientist users from other fields. A highly selected bibliography would have taken only a few pages and would have been an asset. More careful editing could have eliminated confusion arising from statements such as the following a few lines apart: "The number of foreign-born persons is still quite large . . ." and "The fact that the population no longer contains large numbers of foreign-born persons . . ." (p. 348).

Two pleas by Bogue deserve mention. In the chapter on distribution by economic areas and again in the internal migration chapter he advocates greater use of the socially and economically defined system of state economic areas as units for demographic analysis in lieu of the politically defined system of states and census regions. The reviewer agrees that for many purposes the economic areas are clearly superior. Bogue also calls for more attention to detailed occupational and industrial categories instead of the traditional eleven and twelve category classifications. In view of the importance of occupation as a social and economic variable, the reviewer agrees and hopes that attention will be directed to the lengthy tabular supplements to the occupation and industry chapters noted above.

The few criticisms mentioned are minor relative to the breadth of coverage, the impressive compilation of data, and the over-all high quality of the book. The reviewer hopes that the price will not prevent demographers, students, and laymen in applied fields from making extensive use of this outstanding work.

Catalytic Models in Epidemiology. *Hugo Muench.* Cambridge, Mass.: Harvard University Press, 1959. Pp. xiv, 110. $4.50.

MARGARET MARTIN, *Johns Hopkins University*

THE author develops simple mathematical models for expressing two types of relationship in the field of epidemiology. One relationship is that of the proportion by age of individuals showing evidence (by medical history or laboratory test, or both) of having had, at some time in the past, effective contact with a given infectious agent. An underlying assumption of these models is the existence of a "force of infection" (number of effective contacts per person per year) which remains constant over time and for all ages within the range considered, or which is a specified function of age. The "force of infection" then becomes a meaningful constant in the equation. The models apply to the situation where one is taking a broad view of the occurrence of infection in moderately large populations over fairly long periods of time, ignoring fluctuations occurring over short intervals and between age groups. Much of the author's excellent discussion of the advantages and limitations of this simplified approach applies to model-building in general.

The other relationship, treated in a single chapter, is that of the survivorship ratio l_x/l_y for comparing survival of individuals with a chronic condition to that of a life-table population of similar composition, over a time period after the date of diagnosis, say. Here an "increased force of mortality" analogous to the "force of infection" above is postulated.

The equations obtained are those of the simple catalytic (exponential) curve, with fixed and with movable asymptote, together with variations which include the cases of truncated data and of a superimposed epidemic of short duration, the reversible catalytic, and the two-stage catalytic.

The method of fitting is that of moments, using the zeroth and first moments of the distribution by age, or time, with frequencies proportional to the per cent positive in each age group. Rapid graphic estimates of the constants for all the models are obtained by the ingenious use of three simple nomographs.

The assumptions underlying each of the models are clearly stated, and the fact that some of them may be rather unrealistic is discussed. In particular, the data on yaws and on histoplasmin sensitivity do not seem to satisfy very well the assumptions of the two-stage catalytic model, which they are used to illustrate. The usefulness of studying departures from the model for clues to the phenomenon under study is pointed out and illustrated.

More discussion could well have been given to the limitations of the method of fitting used and to the possibility of using alternative methods even though presentation of the latter is not within the scope of the book. The beginner may not recognize, for example, that percentages in age groups where the number examined is small may have undue influence on the constants of the equation. For situations where more precise estimates are desired, together with standard errors, one would feel justified in recommending the method of least squares.

The following details might be mentioned:

On page 33, it may not be immediately apparent to the student that k equals the ratios of observed to calculated values of $\Sigma' A$ as well as of $\Sigma'' A$.

On page 37, the right hand form of equation 4.13 would seem a more direct approach than the form preceding it.

On page 74, last sentence, it would seem that the word *rates* rather than *ratios* is intended.

On pages 91 and 93, the expression "confidence limits" is used incorrectly.

On page 93, second paragraph, the chief disadvantage of wide age grouping is not the one stated.

A few of the statements in Chapter 8 (top of p. 91 and middle of p. 94) do not seem to be in line with the best current and, it is hoped, future practice in sampling.

In summary, the reviewer feels that this book gives the student an interesting and understandable introduction to model-building which is simple enough to encourage him to try his own hand at the game. She has reservations concerning some of the assumptions underlying the models, as does the author, and about the method of fitting.

Regression Analysis. *E. J. Williams.* New York: John Wiley and Sons, 1959. Pp. x, 214.

JACK KIEFER, *Cornell University*

THIS book offers the practical statistician a compendium of the classical techniques associated with regression analysis. The author has included references from recent literature, but the point of view is that which existed before the advent of statistical decision theory. The book is full of good numerical examples which also serve to illustrate the way in which departures from the usual simple models arise in practice. In particular, there are good examples of such topics as estimation from nonlinear equations, estimation of zeros of a fitted curve, etc. A great many of the examples are taken from the field of forestry.

The contents are: Introduction, Linear Regression, Multiple and Polynomial Regression, Regression Equations Requiring Iterative Calculation, Choice Among Regression Formulas, Estimation from the Regression Equation, The Analysis of Covariance, The Treatment of Heterogeneous Data, Simultaneous Regression Equations, Discriminant Functions, Functional Relations, References, Index.

The New Inflation. *Willard L. Thorp* and *Richard E. Quandt.* New York: McGraw-Hill, 1959. Pp. xi, 233. $5.00.

KENNETH E. BOULDING, *University College of The West Indies*

THIS book evidently emerged as a result of a conference of twenty-three economists at the Merrill Center for Economics in July 1958. It is not, however, intended as a summary of the discussions, but is the sole responsibility of the authors. Its attention is focussed on the experience of the postwar years. The title seems to imply that this "new" inflation is somehow different from the old—there are no very startling conclusions, however, and the general style of the work is somewhat textbookish. More attention is paid to structural pressures, and what has elsewhere been called "cost-push" inflation than would have been the case twenty-five years ago. It is perhaps a tribute to the relative stability of economics that there are so few surprises in this authoritative and competent treatment. I must confess that I found the book judicious to the point of dullness. The truth seems to be that economics has reached a certain plateau of analytical competence in regard to this question, which this book faithfully reflects.

One suspects, perhaps, that this is not the final word and that there are mysteries yet to unravel. One would like to know more, for instance, as to the role of expectation and disappointment in the general dynamic process. One would like to know more also about the subtle processes by which information and images are transmitted through the economic system. A complete study of inflation, for instance, would almost have to have a chapter on fashions, moods and epidemics of belief. These higher flights of social science, however, are hardly to be expected from a meeting of twenty-three economists; one wonders, however, if the mixture had included a few non-economists whether something a little more exciting might not have come out of it. However, it is perhaps a little ungracious to judge a book for what it does not pretend to be; for what it is this is an admirable statement of the present state of mind on the

question of inflation among respectable economists, and the authors are to be congratulated on an unusually lucid exposition.

The Theory of Marketing in Underdeveloped Countries. *J. N. Chaturvedi.* Allahabad, India: Kitab Mahal, Publishers, 1959. Pp. 135. $2.00.

LINCOLN CLARK, *New York University*

THE best characteristic of this little book is its intriguing title and it merits commendation for focusing attention on a very important problem. It is the author's response to Galbraith's criticism of Indian economists for not developing "what could be termed an Indian Economic Philosophy."

Chaturvedi's theory stems from his contention that Western economic theories, being based on conditions in rich countries, do not apply to poor countries. His theory starts by distinguishing between "real marketable surplus" and "forced market surplus." The former is that portion of output freely released by a producer; the latter is what he is forced to turn over to his creditors. For the rich (U. S.) farmer, the ratio of real to forced marketable surplus is very high, for the poor (Indian) farmer the ratio is exceedingly low. When this ratio is very low, middlemen are not beneficial agencies resulting from a division of labor but they are, in fact, only extortionate agencies. The solution, therefore, is to supplant them. Grain banks would be established in every village for the storage of all locally produced and consumed produce. These banks would issue (without charge) certificates for all deposited produce. District grain banks would maintain records of the amounts on deposit in the village banks and have equipment for transporting produce. Consumers would pay in advance 20–25% of their monthly income to special retailing societies. These payments would be routed to the grain banks for making advances to the farmers. Concurrently, these retail societies, located principally in urban areas, would collect information about demand from consumers, mills, exporters, etc. and transmit the information to district offices (presumably the district grain banks). The state office would equate and direct the district demands and supplies. This rational scheme would substantially cut distribution costs and, since a very high percentage of the national income of poor countries goes for food, it would improve the national welfare. End of theory.

Even conceding that a politically acceptable method were promulgated for price-setting in order to eliminate the extortionate profits of middlemen, application of the Chaturvedi theory would encounter great cost problems, particularly because it would require a great increase in the ratio of fixed to variable costs. India, for example, has almost countless farmers, villages, wholesalers, retailers, and consumers, most of them operating on a pitifully small scale (and in this reviewer's opinion, with extreme efficiency in their handling of small heterogeneous outputs and minimum capital equipment). The *quick*, in contrast to the *slow* introduction of very large scale market institutions would likely incur much higher costs in collecting, transporting, sorting, warehousing, and retailing the extremely vast quantities of very small heterogeneous outputs.

The book presents ample data and convincing arguments that demonstrate that it is harder for a country to develop if it is poor than if it is rich. Our hearts can bleed for the plight of the poor, and we can sympathize with an innovator's search for a massive quick remedy. In the very old, very populous and very poor countries, however, less dramatic but more effective, is likely to be the work of hundreds of innovators influencing the actions of thousands of market institutions to change the consumption pattern of millions of consumers. The poor countries are going to accelerate the pace of economic development and are prepared to take drastic measures. The rich countries can influence the choice—and one way is to help produce a theory of efficient

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market development. Chaturvedi's book does reinforce the view that a larger scale of operations of particular market institutions is the key. Perhaps this is the clue to the book editor's question, "what is the significance of this book for statisticians?". It seems to call for empirical data arranged in frequency distributions of the size of various types of market transactions. Such data, even without correlates, should help give rise to sound hypotheses for increasing the scale of operations.

Statistics for Business Decisions. *Kurnow, Glasser,* and *Ottman.* Homewood, Illinois: Richard D. Irwin, 1959. Pp. xi, 523. $7.25.

Workbook to Accompany Statistics for Business Decisions. *Kurnow, Glasser,* and *Ottman.* Homewood, Illinois: Richard D. Irwin, 1959. Pp. 189. $3.95.

DELBERT C. HASTINGS, *University of Minnesota*

THIS book represents a partial step away from the conventional "compendium of methods" business statistics text toward the idea of statistics as a tool in business decision-making. The step is only partial, however; the book is by no means a rigorous or even a general introduction to decision theory but is perhaps more aptly described as a recasting of conventional business statistics content in the direction of decision theory. This comment is not intended as criticism; the direction of change is probably desirable and the book may facilitate a change in content of the typical business statistics course which is likely to be gradual in any event.

The scope of coverage of the book is slightly broader than comparable older books; elementary design of experiments, sequential sampling, control charts, for example, are introduced. Otherwise the instructor using the text need be prepared only for a slightly different order of presentation and slightly different emphasis from the usual text.

A unique feature of the accompanying workbook is that all the problems and questions center about a fictional city called Statistica. This device permits applications to problems of school and local government functions, sample surveys, operating problems of local firms, and marketing and industrial aspects of the community. The technique provides interest and continuity.

Problems and examples seem to be well chosen. One small point of taste: The faddish use of "You should now know that, You should now be able to solve, and You will also find that" in place of "Summary, Exercises, References" quickly becomes tedious if not irritating.

The National Debt Ceiling. *Marshall A. Robinson.* Washington: The Brookings Institution 1959. Pp. vii, 104. $1.50. Paper.

J. FRED WESTON, *University of California, Los Angeles*

THIS monograph is a gem. It is a beautiful demonstration of economic analysis applied to a significant issue of public policy. Statisticians will be interested in this study as an excellent illustration of how economic data can be used to make a dispassionate analysis of an important problem, satisfying the criteria of (1) relevance, (2) honesty and balance, (3) rigor, and (4) good judgment.

In a systematic fashion the national debt ceiling is analyzed under the following points: (1) origin, (2) characteristics, (3) the case for it, (4) the case against it, (5) the economics of national debt, and (6) proposal for fiscal discipline. The legal limit on federal borrowing was first placed in a borrowing act of 1917. In 1941 the debt limit in its present form was established when Congress grouped all forms of debt into one figure. The only change in the debt ceiling between 1941 and the time the present book was written was an increase in the ceiling from $65 billion to $288 billion.

The federal debt ceiling reflects the almost irrational fear of debt in general. The

main argument for the ceiling is that it can be used as a device for controlling government spending and to control the growth of the federal debt. Other arguments presented by the advocates of the debt ceiling are that: (1) the debt level is too high, (2) too costly, (3) a major source of inflation, and (4) it threatens national solvency. Mr. Robinson cites the names and the publications in which arguments of these kinds have been presented.

The case against the debt ceiling begins with an analysis of the economics of the national debt. To begin, it is important to recognize that the major increases in the national debt have come during periods of active war such as World War I, World War II, and Korea. Robinson demonstrates that a rising debt during peacetime need not be inflationary; in fact, failure of government debt to grow appropriately can stunt economic growth. Furthermore, increases and decreases in the national debt can be useful in anti-cyclical economic policy. He assembles data to demonstrate that the interest costs of federal debt as a percentage of gross national product have been unchanged since 1945. Furthermore, no great change has developed since 1920. He points out that most of the federal debt is held internally rather than externally and that the transfer effects of debt payments have probably had little influence on either the distribution or level of national income.

In view of the proper role which a normally growing federal debt has in the development of the economy, the harmful effects of the debt ceiling are both undesirable and unnecessary. The kinds of mischief that may be caused by the debt ceiling are seen in sharp perspective in the summer and fall of 1957, when due to the limits of the debt ceiling, the Air Force ceased to pay its bills. The impact of these limitations caused a cut-back in defense spending on the order of magnitude of $8 billion at an annual rate. Coming at a time when the business upswing was already waning, the sharp reversal in spending was sufficient to trigger the down-turn of 1957–8. Despite the cut-back in spending imposed by the debt ceiling, the ensuing recession resulted in a federal deficit of $12.4 billion during fiscal 1959, representing the largest peacetime deficit the country has ever incurred. Thus the irony of the debt ceiling is that it has caused large increases in the national debt, in turn requiring elevations in the permitted ceiling.

Not only may the debt ceiling do the opposite of what it is intended to do, a second major criticism is that it leads to fiscal subterfuge. It causes the federal authorities to use costly and wasteful means to finance programs that Congress has approved. Robinson gives clear illustrations of this point.

Robinson points out that Congress can make better progress toward fiscal discipline by replacing the debt ceiling with budgetary reforms that force Congress to take a comprehensive look at appropriation bills in relation to the revenue system. He mounts a persuasive case that rules are no substitute for understanding. Bad rules are especially pernicious. Robinson demonstrates that the debt ceiling represents the substitution of ritual for intelligence and study. It is a device which has been used as an evasion of true fiscal responsibility. The Brookings Institution performs a most valuable public service by the support of studies of this kind and caliber.

The International Flow of Private Capital, 1956–1958. *United Nations, Department of Economic and Social Affairs.* New York: 1959. Pp. 107. $.75.

ARTHUR I. BLOOMFIELD, *University of Pennsylvania*

THIS report, an outgrowth of the continuing study by the United Nations in the field of foreign investment, is a detailed factual and statistical account of the volume, form, and direction of private international long-term capital movements in recent years, with special emphasis on the period 1956–8. While the focus is pre-

dominantly on direct and portfolio investments, some brief consideration is also given to medium- and short-term international banking credits. The authors have carefully and competently assembled in convenient form, including many useful tables, a mass of relevant data drawn from official statistics and other sources and from the recent experiences of the leading capital-exporting and capital-importing countries. The study, however, contains virtually no analysis, makes no attempt to relate its subject-matter to larger current international economic problems, and avoids specific policy prescriptions. Its brief conclusions follow directly from its factual presentation and are not of a controversial sort.

Among the most important and encouraging developments in this field during the past few years, according to the report, have been a marked increase in the outflow of private long-term capital from Western Europe, especially the United Kingdom and West Germany; a notable revival of portfolio investments, particularly new flotations for foreign account; an increasing diversification of the forms of direct investments, despite continuing concentration in extractive industries; and a higher over-all flow of private foreign investment—the average annual outflow of $4 billion for the period 1955–8 being about twice that in 1951–2. Indeed, the report states that international investments have in recent years entered a new phase and that the prospects for a greater flow of private foreign capital and for a greater contribution of this capital to economic development appear better than ever before.

On the other hand, attention is called to the disturbing fact that countries with the lowest incomes and with no important natural resource endowment, especially in Africa, Asia, and the Far East, where domestic capital formation can hardly keep pace with population growth, still continue to be unsuccessful in attracting any significant volume of private foreign capital. To some degree, official grants and loans have corrected this unevenness in the pattern of private foreign investment. But the major conclusion drawn is the desirability of increasing the contribution that private foreign capital can make to capital formation in the less developed countries. On this matter, however, the report does not outline any measures whereby such an objective might be realized.

Limited though the scope of the study is, it should provide a highly useful source of reference to any student working in the area of current foreign investment problems.

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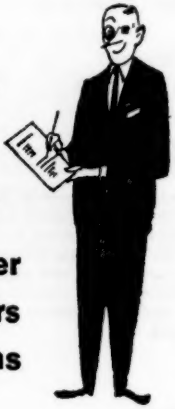

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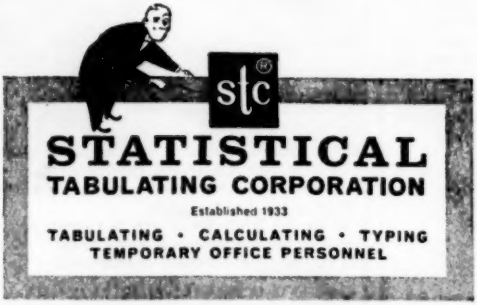

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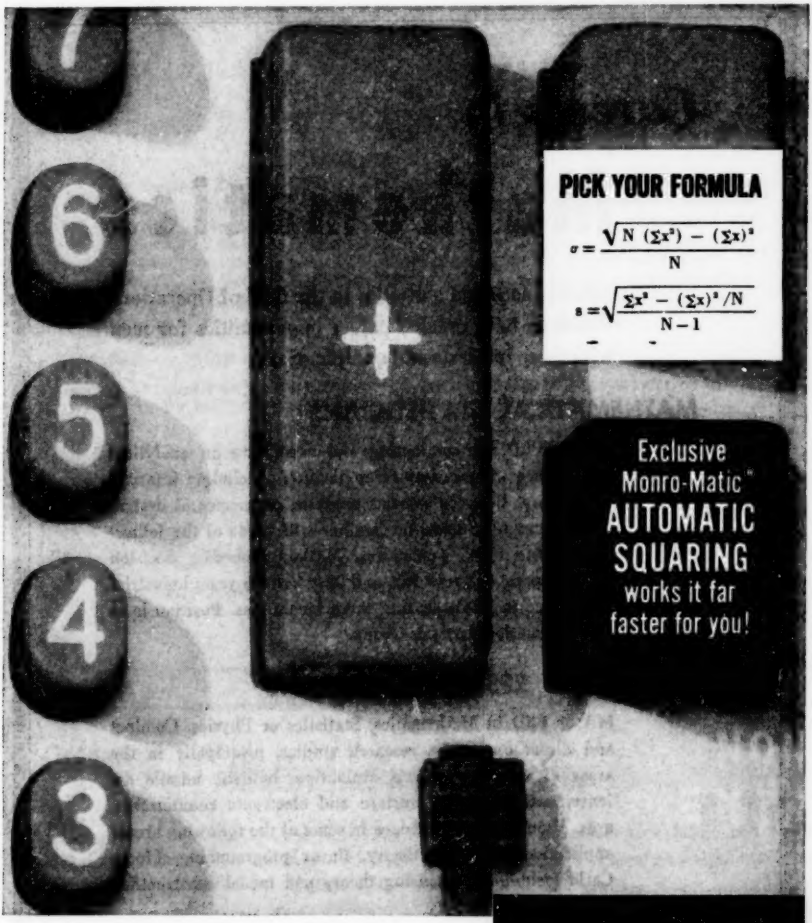

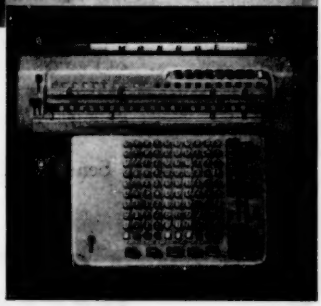

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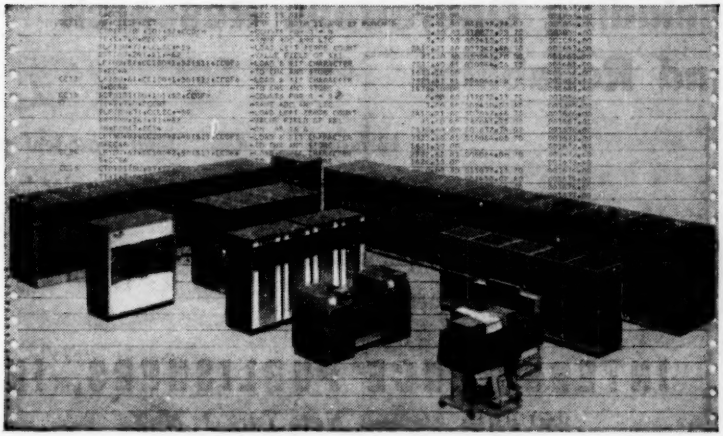

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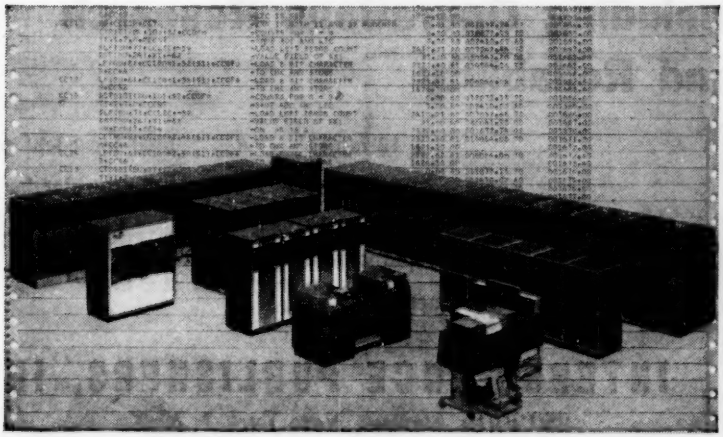

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